Algebraic structures on singular (co)chains

Jonathan R. Potts

A thesis submitted for the degree of

Doctor of Philosophy

in the University of Warwick

July 2006

Contents

Acknowledgements

Firstly, I would like to thank my supervisor, John Jones, for all his help and enthusiasm. I must also thank Yvonne Collins, John's PA, for managing to fit me in to John's hectic schedule! I thank the EPSRC for funding my research. I also thank Warwick University Maths Department and the MEI Further Maths Network for providing me with work to help supplement my grant. On a more personal note, I'd like to thank my parents for their ever-present support and encouragement throughout the last four years (and indeed the twenty-three years prior to that). I'd also like to thank various friends (too many to list) for providing me with frequent and much-needed distractions from my studies. Finally, I would like to thank my wife, Anna, for putting up with my strange mathematical obsessions.

Declaration

I hereby declare that, to the best of my knowledge, the material contained in this thesis is original and my own work, except where otherwise indicated, cited, or commonly-known.

I have not submitted any of this material in partial or complete fulfilment of the requirements for another degree at this or any other university.

Chapter 1

Introduction

Algebraic structures on the singular cochain complex of a space have frequently been used to extract homotopy-theoretic properties of the space. Possibly the best example of this is in the subject known as 'rational homotopy theory'. This began with Quillen in [27]. It deals with homotopy theory when the coefficients are in a field of characteristic 0. In [32], Sullivan developed the subject further by showing that the singular cochain complex with characteristic 0 coefficients is quasi-isomorphic to a commutative differential graded algebra. In the first part of this thesis (chapter 2), we look at an application of this idea to 'string homology' which was first studied in [5] (see also [4]). In the second part (chapter 3), we give a new phrasing of the central idea of rational homotopy theory: the idea that there is a correspondence between the Lie algebra structure on rational homotopy and the commutative algebra structure on rational cochains. In the third part (chapters 4-7), we make some investigations into algebraic structures on the singular (co)chain complex with coefficients in other rings. I give three introductions here to reflect this trichotomy.

1.1 Rational string homology

In [5], Chas and Sullivan discovered (amongst other things) that the 'shifted-by-d' homology of the free loop space of a d-dimensional simply-connected manifold M has the structure of a Gerstenhaber algebra. More precisely, denote the free loop space of a manifold M by $LM := C^\infty(S^1, M)$, the space of smooth loops in M. Denote $\mathbb{H}_n(LM) := H_{d-n}(LM)$ where H_{d-n} is the $(d-n)$-th homology group of LM. Note that homology is concentrated in negative degrees since the differentials in chapter 2 always *increase* degree by 1. There are maps

$$m : \mathbb{H}_*(LM) \times \mathbb{H}_*(LM) \to \mathbb{H}_*(LM)$$

$$[-,-] : \mathbb{H}_*(LM) \times \mathbb{H}_*(LM) \to \mathbb{H}_{*-1}(LM),$$

such that m defines a (degree preserving) commutative multiplication and $[-,-]$ a (degree -1) Lie bracket. The products m and $[-,-]$ are related by the *Gerstenhaber relation*

$$[u, m(v,w)] = m([u,v],w) + (-1)^{|v|(|u|+1)} m(v,[u,w]).$$

($|x|$ denotes the degree of x). This whole structure is called the *string homology of* M. In [2], Cohen and Jones proved that $\mathbb{H}_*(LM;K)$ (the string homology of M with coefficients in a field K) is isomorphic as a Gerstenhaber algebra to the "Hochschild cohomology of the cochains on M", $HH^*(S^*(M;K), S^*(M;K))$, where $S^*(M;K)$ is the cochain algebra of M with coefficients in K. This was contemporaneously proved by Tradler in [33].

In chapter 2, we restrict our attention to the case where the ground field, K, is of characteristic zero (hence "rational" string homology). If this is the case then Sullivan showed (in [32]) that the non-commutative algebra $S^*(M;K)$ is quasi-isomorphic as an associative algebra to an *almost-free* commutative differential graded algebra A. That is, as a graded algebra, A is the free commutative algebra $F_C(V)$ generated by a graded vector space V. We write $A = (F_C(V), \delta)$ where δ is the differential on A. Therefore the central result of chapter 2 is a computation of the Hochschild cohomology of an almost-free differential graded commutative algebra. Specifically, we prove (theorem 2.3.1)

Theorem. *Let V be a finite dimensional graded vector space over a field K of characteristic zero. Let A be the differential graded commutative algebra $(F_C(V), \delta)$ where δ is a differential on $F_C(V)$. Then there is an isomorphism of graded commutative algebras*

$$\mathrm{HH}^*(A, A) \cong H(F_C(\Sigma V^*) \otimes F_C(V), \partial),$$

where ∂ is a functorial construction from δ.

In chapter 2, we use the above theorem to compute the string homology algebras of various manifolds. Specifically, we compute the rational string homology algebras of spheres (of dimensions > 1), complex projective spaces, Grassmannian and Stiefel manifolds. Note that the string homology algebras of spheres and complex projective spaces were calculated in [3]. However, the methods employed in that paper were different to the ones used here. Also [8] contains the results of [3], proved in a different way still. It turns out that the authors of [8] were thinking along similar lines to mine and contemporaneously produced similar results to mine in [9].

Chapter 2 also explains how to compute the Gerstenhaber bracket on $(F_C(V) \otimes F_C(\Sigma V^*), \partial)$ induced by the isomorphism of theorem 2.2.1. The bracket is computed explicitly in the cases where M is an even dimensional sphere or complex projective space.

1.2 Rational cochains and rational homotopy

Fix a field K of characteristic 0 and a topological space X. As mentioned previously, Sullivan showed in [32] that the rational cochain algebra $S^*(X; K)$ is quasi-isomorphic to an almost-free commutative differential graded algebra $(F_C(V), \delta)$. In addition, he showed that V is isomorphic to $\mathrm{Hom}(\pi_*(X), K)$. This makes $\Sigma\pi_*(X) \otimes K$ into an L_∞-algebra (by definition - see definition 3.3.2). The quadratic part of this L_∞-algebra structure is the Lie algebra structure given by the Whitehead product (see [7, proposition 13.16]). In chapter 3, we describe the rest of this L_∞ structure in terms of 'higher Whitehead products' on $\Sigma\pi_*(X) \otimes K$, which we introduce.

There is a natural cohomology theory for L_∞-algebras, denoted $H^*(A; L_\infty)$ when A is an L_∞-algebra. We use this to prove the following (see theorem 3.4.1).

Theorem. *There is an L_∞-structure on $\Sigma\pi_*(X)$ such that:*

(i) The linear part is zero.

(ii) The quadratic part is dual to the Whitehead product.

(iii) There is an isomorphism of graded commutative algebras

$$H^*(\Sigma\pi_*(X); L_\infty) \cong H^*(X).$$

There is also a natural cohomology theory for C_∞-algebras, denoted $H^*(A; C_\infty)$ when A is a C_∞-algebra. We use this to prove the following (see theorem 3.3.1).

Theorem. *There is a C_∞-structure on the reduced cohomology $\bar{H}^*(X)$ such that:*

(i) The linear part is zero.

(ii) The quadratic part is dual to the cup product.

(iii) There is an isomorphism of graded Lie algebras

$$H^*(\bar{H}^*(X); C_\infty) \cong \Sigma\pi_*(X).$$

These two theorems give us a new phrasing of the central Lie-commutative duality in rational homotopy theory.

1.3 Cochains over other rings

In chapter 4, we move our attention from fields of characteristic zero to those of characteristic p for some prime number p. Let X be a CW complex and $\mathbb{F}$ a field of characteristic p. The complex $S^*(X;\mathbb{F})$ is not in general quasi-isomorphic to a commutative differential graded algebra. However, it is an E_∞-algebra (see [17] for an introduction to E_∞-algebras). It was shown in [13] that $S^*(X;\mathbb{F})$ is an algebra over a particular operad. In [20, section 1], this operad was used to construct an E_∞-operad, $\mathcal{E}$ that also acts on $S^*(X;\mathbb{F})$. Indeed, this structure fixes the weak homotopy type of X under some mild assumptions (see [20], [21]).

In [28], Robinson and Whitehead constructed a homology theory for E_∞-algebras called Γ-*homology*. The Γ-homology of an E_∞-algebra A is denoted $\Gamma H_*(A)$. The functor $\Gamma H_*(-)$ is the homology of an appropriate indecomposables functor for E_∞-algebras. The following result is proved in chapter 4 (theorem 4.0.1):

Theorem. *Let $\mathbb{F}$ be a field of characteristic p. Let X be a path-connected p-complete nilpotent space of finite p-type. Then*

$$\Gamma H_*(S^*(X;\mathbb{F})) \cong 0.$$

This result arises from the fact that the Steenrod operator P^0 is the identity on $H^*(X;\mathbb{F})$ but is not the identity on every E_∞-algebra. Hence every element of $H^*(X;\mathbb{F})$ that represents a non-trivial homology class is homologous to $P^0(x)$ for some $x \in H^*(X;\mathbb{F})$, so is decomposable. The theorem is proved by translating this idea to the level of cochains.

One of the problems with dealing with E_∞-operads is that there was, until now, no combinatorial example of an E_∞-operad analogous to the operad of associahedra for A_∞-operads (see [24, sections I.1.6]). This problem was noted in [24, sections I.1.11].

In chapter 5, this is addressed by introducing a new E_∞-operad called the *step operad*, $\mathcal{S}$. It is constructed using the ideas of [30] and acts naturally on the singular (co)chains of a topological space X. In chapter 5, we restrict our attention to the case where the ground field has characteristic 2. For this case we prove (theorem 5.5.1):

Theorem. $S_*(X)$ *is a coalgebra over* $\mathcal{S}$.

In chapter 6, we extend our definition of $\mathcal{S}$ to an arbitrary commutative unital ground ring R. Specifically, we define an operad $\mathcal{S}_R$ and show that $S_*(X;R)$ is a coalgebra over $\mathcal{S}_R$. The quadratic suboperad $\mathcal{Q}_R$ of $\mathcal{S}_R$ is also defined. $\mathcal{Q}_R$ is an operad that encodes the Steenrod cup-i products, their iterations and nothing more. Using this and [21, main theorem] the following theorem is proved (theorem 6.5.2):

Theorem. *Let X and Y be finite type nilpotent spaces. Then X and Y are weakly equivalent if and only if there is a quasi-isomorphism $S^*(X;\mathbb{Z}) \to S^*(Y;\mathbb{Z})$ that commutes with all the cup-i products.*

In fact, I suspect that $\mathcal{S}_R = \mathcal{Q}_R$, but fell short of being able to prove this.

As has been mentioned, E_∞-operads that act on $S^*(X)$ have been constructed before; namely the operad $\mathcal{E}$ mentioned previously and an operad defined by Justin Smith in [29], denoted $\mathcal{G}$. However, both constructions are quite different to $\mathcal{S}$. Particularly, neither are constructed in such a combinatorial fashion.

Let $\mathbb{F}$ be a field. In chapter 7, we make some initial efforts into constructing a functor that takes the (co)chains on a space to the homotopy groups of the space. Specifically we define 'space-like' $\mathcal{S}_\mathbb{F}$-coalgebras and a collection of functors $\pi_n(-)$ for positive integers n that take space-like $\mathcal{S}_\mathbb{F}$-coalgebras to groups. Denoting the homotopy groups of a topological space X by $\pi_n(X)$ as usual (the ambiguity being cleared up by the context) the following theorems are proved (theorems 7.0.6 and 7.0.7):

Theorem. *For a connected topological space X, the complex of singular chains $S_*(X;\mathbb{F})$ is a space-like $\mathcal{S}_\mathbb{F}$-coalgebra and $\pi_n(S_*(X;\mathbb{F})) \cong \pi_n(X)$.*

Theorem. *Let C, D be space-like $\mathcal{S}_\mathbb{F}$-coalgebras. Suppose there is a quasi-isomorphism of $\mathcal{S}_\mathbb{F}$-coalgebras $f : C \to D$. Then $\pi_n(f) : \pi_n(C) \to \pi_n(D)$ is injective.*

It does not appear to be true that $\pi_n(f)$ is necessarily an isomorphism when f is a quasi-isomorphism. However, this is in some ways unsurprising, since $\pi_n(-)$ does not refer to any closed model structure on space-like $\mathcal{S}_{\mathbb{F}}$-coalgebras. One would expect that in a 'correct' definition of $\pi_n(-)$ (that is, one that takes quasi-isomorphisms to isomorphisms), $\pi_n(-)$ would be the *derived* functor of some functor or other. Therefore one would need to prove that quasi-isomorphisms of *cofibrant* objects are sent to isomorphisms of groups. As it stands, however, $\pi_n(-)$ treats all space-like $\mathcal{S}_{\mathbb{F}}$-coalgebras on the same footing, not caring whether the object is cofibrant or not. The reason for constructing $\pi_n(-)$ in this fashion is because it proved tricky to pin down a sensible notion of 'cofibrant space-like $\mathcal{S}_{\mathbb{F}}$-coalgebras'. Nonetheless, I expect such a notion exists and would lead to a better definition of $\pi_n(-)$.

Chapter 2

Rational string homology

2.1 Signs, grading and notation

Throughout the course of this chapter, the following sign convention will be in place: whenever two objects of degree n and m are permuted, the sign $(-1)^{nm}$ is accrued. The degree of an element x will be denoted $|x|$. For example, for a degree 1 derivation δ,

$$\delta(xy) = \delta(x)y + (-1)^{|\delta||x|}x\delta(y).$$

Further, everything will be $\mathbb{Z}$-graded. Thus by *commutative*, we mean *graded commutative* so that $xy = (-1)^{|x||y|}yx$; whenever we talk about a Lie algebra, we have that $[x, y] = -(-1)^{|x||y|}[y, x]$ etc.

All boundary maps will raise degree by 1. This may affect the "natural" grading of certain objects. For example, the homology and homotopy groups of spaces will be concentrated in negative degrees to allow for this convention.

For an integer n and a differential graded vector space V, we use V_n to denote the n-th degree part of V. Frequently, we will wish to shift the degrees of V. We use the sign Σ to do this by defining the m-th degree part of the n-th suspension of V to be the $(m-n)$-th degree part of V:

$$(\Sigma^n V)_m := V_{m-n}.$$

For elements of V, we use a lower case sigma to do the same job. In other words $v \in V$ if and only if $\sigma^n(v) \in \Sigma^n(V)$. As $\sigma^{-1}v$ is often used in the calculations of chapter 2, we denote $\alpha := \sigma^{-1}$. ΣV^* always means the suspension of the dual of V (as opposed to the dual of the suspension).

Elements of graded vector spaces will have subscripts displaying their degree if needed (e.g. u_n is an element in degree n). If V is a graded vector space with basis X and $x \in X$ then we use x^* to denote the unique element in V^* such that $x^*(x) = 1$ and $x^*(y) = 0$ for each $y \in X$ with $x \neq y$.

The degree of a map is the amount the map *raises* degree by. If R is a ring and C, D are graded R-modules then $\mathrm{Hom}^n_R(C, D)$ denotes the graded module of sequences of homomorphisms over R from C_m to D_{n+m} for each m.

$F_C(V)$ is used to denote the free graded commutative (unital) algebra generated by a graded vector space V. TV is used to denote the free associative algebra (tensor algebra) generated by V. If $v_{n_1}, \ldots, v_{n_k} \in V$ then $v_{n_1} \otimes \cdots \otimes v_{n_k} \in TV$ has degree $n_1 + \cdots + n_k$. $V^{\otimes n}$ means the n-fold tensor product $V \otimes \cdots \otimes V$.

2.2 The Hochschild cohomology of a differential graded algebra

In this section, we summarise some background results that are used in this chapter. For a more detailed exposition, see for example [18] or [14].

Let (A, δ) be a differential graded algebra. Define a double complex $\{M^{p,q}, b, d\}$ where

$$M^{p,q} = \mathrm{Hom}_K^{p-1}((\Sigma^{-1}A)^{\otimes q}, \Sigma^{-1}A),$$

b is the Hochschild boundary map and d is the map $f \mapsto \delta \circ f + (-1)^{|f|} f \circ \sum_i 1^{\otimes i} \otimes \delta \otimes 1^{\otimes q-i-1}$. The *Hochschild cochain complex of* A is the total complex of this double complex. It is denoted $\mathrm{C}^*(A, A)$. The *Hochschild cohomology of* A is the homology of $\mathrm{C}^*(A, A)$. It is denoted $\mathrm{HH}^*(A, A)$.

Let $m \in \mathrm{Hom}^1(\Sigma^{-1}A \otimes \Sigma^{-1}A, \Sigma^{-1}A)$ be the element induced by the multiplication map in A. The Hochschild cochain complex can be described as a cosimplicial chain complex with the following codegeneracy and coface maps:

$$s_i : \mathrm{C}^{n+1}(F_C(V), F_C(V)) \to \mathrm{C}^n(F_C(V), F_C(V))$$

$$d_i : \mathrm{C}^{n-1}(F_C(V), F_C(V)) \to \mathrm{C}^n(F_C(V), F_C(V))$$

$$s_i(f_{n+1})(\alpha(a_1)\otimes\cdots\otimes\alpha(a_n)) = f_{n+1}(\alpha(a_1)\otimes\cdots\otimes\alpha(a_i)\otimes 1\otimes\alpha(a_{i+1})\otimes\cdots\otimes\alpha(a_n))$$

for every $i \in \{0, \ldots, n\}$,

$$d_0(f_{n-1}) = m(1 \otimes f_{n-1}),$$

$$d_i(f_{n-1}) = f_{n-1}(1^{\otimes i-1} \otimes m \otimes 1^{\otimes n-i-1})$$

for every $i \in \{1, \ldots, n-1\}$ and

$$d_n(f_{n-1}) = m(f_{n-1} \otimes 1).$$

The *reduced Hochschild cochain complex*, $\mathrm{NC}^*(A, A)$, is the sub-complex of $\mathrm{C}^*(A, A)$ consisting of those maps that take every element in $\{\alpha(a_1) \otimes \cdots \otimes \alpha(a_q) : a_i \in K \textit{ for some } i\}$ to zero. As is standard (see for example [22, section 22]) the inclusion $i : \mathrm{NC}^*(A, A) \to \mathrm{C}^*(A, A)$ is a quasi-isomorphism of differential graded vector spaces.

The following (standard) isomorphism is used in this chapter (see for example [18, page 37ff] for a proof):

$$\mathrm{HH}^*(A, A) \cong \mathrm{Ext}^*_{A \otimes A}(A, A). \tag{2.1}$$

2.3 The main theorem

Let K be a field of characteristic 0. K will be the ground field throughout this chapter.

Definition 2.3.1. Let V be a graded vector space. Let A be a differential graded algebra such that A is $F_C(V)$ as a graded algebra. Suppose that the induced differential on V is zero. Then A is called *an almost free commutative differential graded algebra generated by V.*

Definition 2.3.2. Let S be a differential graded algebra. An *almost free resolution of S* is an almost free commutative differential graded algebra A, generated by a graded vector space V, such that A is quasi-isomorphic to S.

For example, if X is a manifold and $\Omega^*(X)$ is the algebra of forms on X then a minimal model of $\Omega^*(X)$ is an almost free resolution of $\Omega^*(X)$ (minimal models are defined in [6]).

Let X be a path-connected, simply-connected, finite-dimensional manifold. Let A be an almost free resolution of $S^*(X)$. Then there is a differential graded algebra B and quasi-isomorphisms $S^*(X) \to B \leftarrow A$. We have the following isomorphisms of graded algebras:

$$\mathbb{H}_*(LX) \cong \mathrm{HH}^*(S^*(X), S^*(X)) \cong \mathrm{HH}^*(A, A).$$

The first of these is the Cohen-Jones isomorphism. The second is induced by the following sequence of natural quasi-isomorphisms:

$$\mathrm{C}^*(S^*(X), S^*(X)) \to \mathrm{C}^*(S^*(X), B) \leftarrow \mathrm{C}^*(B, B) \to \mathrm{C}^*(A, B) \leftarrow \mathrm{C}^*(A, A).$$

Let V be a finite dimensional graded vector space. Let A be an almost free commutative differential graded algebra generated by V, with differential δ. That is, $A = (F_C(V), \delta)$. Before we state the main theorem of the chapter, we need to demonstrate how δ induces a differential on $F_C(\Sigma V^*) \otimes F_C(V)$.

Pre-composing the inclusion $V \hookrightarrow F_C(V)$ with δ gives a map

$$V \to F_C(V).$$

We compose this map with

$$F_C(V) \to V \otimes F_C(V)$$

$$v_1 \dots v_n \mapsto \sum_{i=1}^{n} v_i \otimes v_1 \dots v_{i-1} v_{i+1} \dots v_n$$

(where $v_1, \dots, v_n \in V$) to give a map $V \to V \otimes F_C(V)$ of degree 1. Dualising and suspending gives a map

$$\Sigma V^* \otimes F_C(V)^* \to \Sigma V^*$$

which, in turn, induces a map

$$\Sigma V^* \to \Sigma V^* \otimes F_C(V)$$

that extends to a derivation

$$d : F_C(\Sigma V^*) \to F_C(\Sigma V^*) \otimes F_C(V).$$

Then the differential

$$\partial : F_C(\Sigma V^*) \otimes F_C(V) \to F_C(\Sigma V^*) \otimes F_C(V)$$

is given by $\partial = d \otimes 1 + 1 \otimes \delta$.

Definition 2.3.3. ∂ is called *the differential on* $F_C(\Sigma V^*) \otimes F_C(V)$ *induced by* δ.

The main theorem of this chapter is:

Theorem 2.3.1. *Let* V *be a finite dimensional graded vector space. Let* A *be an almost free differential graded commutative algebra generated by* V. *Suppose* A *has differential* δ. *Let* ∂ *be the differential on* $F_C(\Sigma V^*) \otimes F_C(V)$ *induced by* δ. *Then there is an isomorphism of graded algebras*

$$\mathrm{HH}^*(A, A) \to H_*(F_C(\Sigma V^*) \otimes F_C(V), \partial).$$

This map is given explicitly as follows. Let X be a basis for V. Let $\theta : \mathrm{C}^*(A, A) \to (F_C(\Sigma V^*) \otimes F_C(V))$ be given by

$$\theta(f) = \sum_{x_1, \ldots, x_n \in X} \alpha(x_1)^* \ldots \alpha(x_n)^* \otimes f(\alpha(x_1) \otimes \cdots \otimes \alpha(x_n))$$

for each map $f : \Sigma^{-1} A^{\otimes n} \to A$. Let θ_* be the map induced on homology by θ.

Theorem 2.3.2. *The isomorphism of theorem 2.3.1 is given by* θ_*.

2.4 Proof of the main theorem

Let V be a finite dimensional graded vector space.

Theorem 2.4.1. *There is an isomorphism of graded algebras*

$$\mathrm{HH}^*(F_C(V), F_C(V)) \xrightarrow{\cong} F_C(\Sigma V^*) \otimes F_C(V).$$

This is a consequence of the following three lemmas:

Lemma 2.4.2. *Let V be a finite dimensional graded vector space generated by one element z of odd degree. Then there is an isomorphism of graded algebras*

$$\mathrm{HH}^*(F_C(V), F_C(V)) \xrightarrow{\cong} F_C(\Sigma V^*) \otimes F_C(V).$$

Lemma 2.4.3. *Let V be a finite dimensional graded vector space generated by one element z of even degree. Then there is an isomorphism of graded algebras*

$$\mathrm{HH}^*(F_C(V), F_C(V)) \xrightarrow{\cong} F_C(\Sigma V^*) \otimes F_C(V).$$

Lemma 2.4.4. *Let U, V be finite dimensional graded vector spaces. Then*

$$\mathrm{HH}^*(F_C(U), F_C(U)) \otimes \mathrm{HH}^*(F_C(V), F_C(V)) \cong \mathrm{HH}^*(F_C(U \oplus V), F_C(U \oplus V)).$$

Proof of lemma 2.4.2. We work with the reduced Hochschild cochain complex, $\mathrm{NC}^*(F_C(V), F_C(V))$. As stated in section 2.2, the inclusion

$$i : \mathrm{NC}^*(F_C(V), F_C(V)) \to \mathrm{C}^*(F_C(V), F_C(V))$$

is a quasi-isomorphism of differential graded vector spaces.

We need to prove that the Hochschild boundary vanishes on restriction to normalisation. Let z be the generator of V and suppose $|z| = 2k+1$. Let $f_n \in \mathrm{NC}^n(F_C(z), F_C(z))$. Note that if $i \in \{1, \ldots, n-1\}$ then $d_i(f_n)(\alpha(z)^{\otimes n+1}) = f_n(\alpha(z) \otimes \cdots \otimes \alpha(z^2) \otimes \cdots \otimes \alpha(z))$, which is zero since z^2 is zero as z is an odd variable. Therefore it is necessary to show that d_0 and d_n either both vanish or cancel one another out.

Suppose f_n is of degree $p-1$. Decompose f_n as $f_n = f_n^0 + \alpha(z) f_n^1$ where

$$f_n^0 : (T\Sigma^{-1} F_C(z))_{-p} \to \Sigma^{-1} K$$

and

$$f_n^1 : (T\Sigma^{-1}F_C(z))_{2k+1-p} \to \Sigma^{-1}K.$$

Any element of $T\Sigma^{-1}F_C(z)$ that is not mapped to zero by f_n must be of even degree, since $|\alpha(z)| = 2k$. Therefore if p is odd then $f_n = \alpha(z)f_n^1$. Likewise, if p is even then $f_n = f_n^0$. Let $k_0, k_1 \in K$ be given by $f_n^0(\alpha(z)^{\otimes n}) = \alpha(k_0)$, $f_n^1(\alpha z^{\otimes n}) = \alpha(k_1)$. Recall that the Hochschild boundary map b is given by the formula $bf = [f, m]$ where m is as defined in section 2.2 and $[-,-]$ is the Gerstenhaber bracket. Then, writing $f = f_n$,

$$bf(\alpha(z)^{\otimes n+1}) = [f,m](\alpha(z)^{\otimes n+1}) = (-1)^{|f|+1} m \circ (f \otimes 1 + 1 \otimes f)(\alpha(z)^{\otimes n+1})$$

$$= (-1)^{|f|+1}\left[(-1)^{|f(\alpha(z)^{\otimes n})|+1} f(\alpha(z)^{\otimes n})\alpha(z) + (-1)^{|f||\alpha(z)|}(-1)^{|z|}\alpha(z)f(\alpha(z)^{\otimes n})\right]$$

$$= \begin{cases} -\alpha((k_1 z)z) - \alpha(z(k_1 z)) = 0 & \text{if } p \text{ is odd} \\ k_0\alpha(z) - k_0\alpha(z) = 0 & \text{if } p \text{ is even} \end{cases}$$

Therefore the boundary map vanishes on normalisation. Let

$$\phi : \mathrm{NC}^*(F_C(V), F_C(V)) \to \mathrm{Hom}_K(F_C(\Sigma^{-1}V), F_C(V))$$

be given by $\phi(f)(\alpha(z)^n) = f(\alpha(z)^{\otimes n})$. Since $\alpha(z)$ is of even degree, it follows from the definition of $\mathrm{NC}^*(F_C(V), F_C(V))$ that ϕ is an isomorphism. There is a canonical isomorphism $c : \mathrm{Hom}_K(F_C(\Sigma^{-1}V), F_C(V)) \to F_C(\Sigma V^*) \otimes F_C(V)$. Thus there are degree preserving quasi-isomorphisms of algebras

$$\mathrm{C}^*(F_C(V), F_C(V)) \xleftarrow{\ i\ } \mathrm{NC}^*(F_C(V), F_C(V)) \xrightarrow{\ c\circ\phi\ } F_C(\Sigma V^*) \otimes F_C(V).$$

These induce an isomorphism in homology, proving the lemma. □

Proof of lemma 2.4.3. In this case the boundary does not vanish on normalisation. We therefore take a different approach (adapted from the proof of [18, theorem 3.2.2]). Recall from section 2.2 that

$$\mathrm{HH}^*(A, A) \cong \mathrm{Ext}^*_{A\otimes A}(A, A).$$

Let $\mathbf{P}_A^+$ be a contractible complex $\cdots \to A \otimes A \to A \to 0$, also known as *a projective $A \otimes A$-resolution of A*. Let $\mathbf{P}_A$ be obtained from $\mathbf{P}_A^+$ by removing the two right-hand terms from the complex. This is called *the deleted projective resolution corresponding to $\mathbf{P}_A^+$*. By definition, $\mathrm{Ext}^*_{A\otimes A}(A, A) = H^*(\mathrm{Hom}_{A\otimes A}(\mathbf{P}_A, A))$. Now let $A = F_C(V)$. Suppose V is generated by an element z of even degree. Let $\mathbf{P}_A^+$ be the resolution

$$0 \to F_C(V) \otimes V \otimes F_C(V) \xrightarrow{\quad\Delta\quad} F_C(V) \otimes F_C(V) \xrightarrow{\quad m\quad} F_C(V) \to 0,$$

where Δ is defined so that $\Delta(x \otimes v \otimes y) = xv \otimes y - x \otimes vy$ for $x, y \in F_C(V)$ and $v \in V$. Let $\mathbf{P}_A$ be the corresponding deleted projective resolution. It follows that $\mathrm{Hom}_{A\otimes A}(\mathbf{P}_A, A)$ is

$$\mathrm{Hom}_K(K, F_C(V)) \xrightarrow{\quad 0\quad} \mathrm{Hom}_K(V, F_C(V)) \to 0.$$

Therefore $H^*(\mathrm{Hom}_{A\otimes A}(\mathbf{P}_A, A)) = \mathrm{Hom}_K(\Sigma^{-1}V, F_C(V)) \oplus \mathrm{Hom}_K(K, F_C(V))$. Now, V is generated by one even degree element so $\Sigma^{-1}V$ is generated by one odd degree element. Therefore $\mathrm{Hom}_K(\Sigma^{-1}V, F_C(V)) \oplus \mathrm{Hom}_K(K, F_C(V))$ is canonically isomorphic to $\mathrm{Hom}_K(F_C(\Sigma^{-1}V), F_C(V))$. This in turn is canonically isomorphic to $F_C(\Sigma V^*) \otimes_K F_C(V)$. In summary,

$$\mathrm{HH}^*(A, A) \cong F_C(\Sigma V^*) \otimes_K F_C(V). \tag{2.2}$$

□

Note. In the proof of lemma 2.4.3, if the basis of V were to contain an odd degree element then the sequence $\mathbf{P}_A^+$ would not be exact. Therefore this argument does not generalise (e.g. if $x \in V$ is an element of odd degree than $x \otimes x \otimes x \in F_C(V) \otimes V \otimes F_C(V)$ is a non-trivial element in the kernel of Δ so Δ is not injective).

Proof of lemma 2.4.4. Observe the following sequence of isomorphisms

$$\mathrm{HH}^*(F_C(U \oplus V), F_C(U \oplus V)) = H^*(\mathrm{C}^*(F_C(U \oplus V), F_C(U \oplus V)))$$

$$\cong H^*(\mathrm{C}^*(F_C(U), F_C(U)) \otimes \mathrm{C}^*(F_C(V), F_C(V)))$$

$$\cong H^*(\mathrm{C}^*(F_C(U), F_C(U))) \otimes H^*(\mathrm{C}^*(F_C(V), F_C(V))).$$

The first isomorphism can be checked directly using basic properties of tensor products. The second isomorphism is the algebraic Künneth isomorphism. □

Proof of theorem 2.3.1. By theorem 2.4.1, we have a map of double complexes

$$\{\mathrm{NC}^*(F_C(V), F_C(V)), d, b\} \to \{F_C(V) \otimes F_C(\Sigma V^*), \partial, 0\}$$

where b is the Hochschild boundary, 0 is the zero differential and

$$df = \delta \circ f + (-1)^{|f|} f \circ \sum_i 1^{\otimes i} \otimes \delta \otimes 1^{\otimes q-i-1}.$$

This gives a map of spectral sequences by taking b and 0 to be the first differentials. Theorem 2.4.1 shows that the morphism induced on E^1-terms is an isomorphism. By [25, theorem 3.4] this induces an isomorphism on the homology of the total complexes. □

Proof of theorem 2.3.2. By lemma 2.4.4, it suffices to examine the cases where V is generated by one element. In the case where V is generated by an odd degree element, it is easy to see that θ restricted to the normalised Hochschild cochains is precisely the map $c \circ \phi$ defined in the proof of 2.4.2. In the case where V is generated by an even degree element, we need to show that θ_* realises isomorphism (2.2). First we need to write down the map that gives isomorphism (2.1). Recall that (2.1) says:

$$\mathrm{HH}^*(A, A) \cong \mathrm{Ext}^*_{A\otimes A}(A, A). \tag{2.1}$$

Now, $\mathrm{Ext}^*_{A\otimes A}(A, A)$ is by definition the homology of $\mathrm{Hom}_{A\otimes A}(\mathbf{P}_A, A)$ where $\mathbf{P}^+_A$ is a projective $A \otimes A$-resolution of A and $\mathbf{P}_A$ is the corresponding deleted projective resolution. Thus we can prove isomorphism (2.1) by taking $\mathbf{P}^+_A$ to be the bar resolution of A. With this resolution, $\mathrm{C}^*(A, A)$ is precisely $\mathrm{Hom}_{A\otimes A}(\mathbf{P}_A, A)$. Consider the commutative diagram:

$$\begin{array}{ccccccccc}
\to & 0 & \to & A\otimes V\otimes A & \xrightarrow{\Delta} & A\otimes A & \xrightarrow{m} & A \\
 & \downarrow{\scriptstyle inc} & & \downarrow{\scriptstyle inc} & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} \\
\to & A\otimes A\otimes A\otimes A & \to & A\otimes A\otimes A & \to & A\otimes A & \to & A
\end{array}$$

where the bottom row is the bar resolution of A. Let $\mathbf{P}^+_A$ denote the $A \otimes A$-resolution of A given by the top line of this diagram and $\mathbf{P}_A$ the corresponding deleted resolution. Then the diagram describes a quasi-isomorphism from $\mathbf{P}^+_A$ to the bar resolution of A. This gives a map $\mathrm{C}^*(A, A) \to \mathrm{Hom}_{A\otimes A}(\mathbf{P}_A, A)$. Composing this with the isomorphism $\mathrm{Hom}_{A\otimes A}(\mathbf{P}_A, A) \to F_C(\Sigma V^*) \otimes_K F_C(V)$ gives a map $\mathrm{C}^*(A, A) \to F_C(\Sigma V^*) \otimes_K F_C(V)$ which is identical to θ. □

2.5 String homology algebra calculations

2.5.1 Stiefel manifolds and odd dimensional spheres

The string homology algebras of Stiefel manifolds and odd dimensional spheres are simple to calculate, since the cochain algebras are quasi-isomorphic to free commutative algebras. So we use the following theorem:

Theorem 2.5.1. *Let X be a finite dimensional simply connected manifold such that $S^*(X)$ is quasi-isomorphic as a differential graded algebra to $F_C(V)$ with zero differential, for some finite dimensional graded vector space V. Then there is a degree-preserving isomorphism of graded algebras*

$$\mathbb{H}_*(LX) \cong F_C(V) \otimes_K F_C(\Sigma V^*).$$

Proof. $\mathbb{H}_*(LX) \cong \mathrm{HH}^*(S^*(X), S^*(X))$ and since $S^*(X)$ is quasi-isomorphic to $F_C(V)$, $\mathrm{HH}^*(S^*(X), S^*(X)) \cong \mathrm{HH}^*(F_C(V), F_C(V))$. □

If X is a *formal manifold* (see [6]) then there exists an algebra B and quasi-isomorphisms $S^*(X) \to B \leftarrow H^*(X)$ which induce an isomorphism $\mathrm{HH}^*(S^*(X), S^*(X)) \cong \mathrm{HH}^*(H^*(X), H^*(X))$.

Since Stiefel manifolds and spheres are formal manifolds (see [6]), the next two corollaries follow immediately from the following two facts (which are well-known: see for example [26]):

$$H^*(V_k(\mathbb{C}^n)) \cong F_C(z_{2(n-k)+1}, z_{2(n-k)+3}, \dots, z_{2n-1})$$

$$H^*(\mathbb{S}^{2k+1}) \cong F_C(z_{2k+1}).$$

Note. Here we are making use of the notation explained in section 2.1.

Corollary 2.5.2. *Let $V_k(\mathbb{C}^n)$ denote the Stiefel manifold of complex k-frames in $\mathbb{C}^n$. Then there is a degree-preserving isomorphism of graded algebras*

$$\mathbb{H}_*(LV_k(\mathbb{C}^n)) \cong F_C(z_{2(n-k)+1}, z_{2(n-k)+3}, \dots, z_{2n-1}, u_{2(k-n)}, u_{2(k-n)-2}, \dots, u_{2-2n}).$$

Corollary 2.5.3. *Let $\mathbb{S}^{2k+1}$ denote the sphere of dimension $2k+1$. Then there is a degree-preserving isomorphism of graded algebras*

$$\mathbb{H}_*(L\mathbb{S}^{2k-1}) \cong F_C(z_{2k+1}, u_{-2k}).$$

2.5.2 Even dimensional spheres.

An almost-free resolution of $S^*(\mathbb{S}^{2k})$ is given by $M = (F_C(u_{2k}, v_{4k-1}), \delta)$ (see, for example [7]) where $\delta(v_{4k-1}) = u_{2k}^2$ and $\delta(u_{2k}) = 0$. Recall the notation used in section 2.3 for the various boundary maps. Theorem 2.3.1 gives the isomorphism

$$\mathbb{H}_*(L\mathbb{S}^{2k}) \cong H_*(F_C(w_{1-2k}, z_{2-4k}, u_{2k}, v_{4k-1}), \partial).$$

More precisely, we calculate ∂ on the generators:

$$\partial(v) = u^2$$

$$\partial(u) = 0.$$

Now we calculate $\partial(w)$ and $\partial(z)$. Note that w is the dual element in ΣV^* to u in V. Therefore $\partial(w)(\alpha(u)) = w \circ \delta(\alpha(u)) = 0$ and $\partial(w)(\alpha(v)) = w \circ \delta(\alpha(v)) = w(\alpha(u^2)) = \alpha(w)(u^2) = u$. Now, z is the dual element in ΣV^* to v in V so $uz(\alpha(u)) = 0$ and $uz(\alpha(v)) = u$. Therefore

$$\partial(w) = uz.$$

Calculating $\partial(z)$ we find $\partial(z)(\alpha(u)) = z \circ \delta(\alpha(u)) = 0$ and $\partial(z)(\alpha(v)) = z \circ \delta(\alpha(v)) = z(\alpha(u^2)) = 0$. Thus

$$\partial(z) = 0.$$

It follows that $\mathbb{H}_*(L\mathbb{S}^{2k})$ contains the algebra

$$\frac{F_C(u,z)}{(u^2, uz)}.$$

It is now necessary to check if any other polynomials survive to homology. We calculate:

$$\partial(uv) = u^3$$

$$\partial(uw) = u^2 z$$

$$\partial(vw) = u^2 w - vuz$$

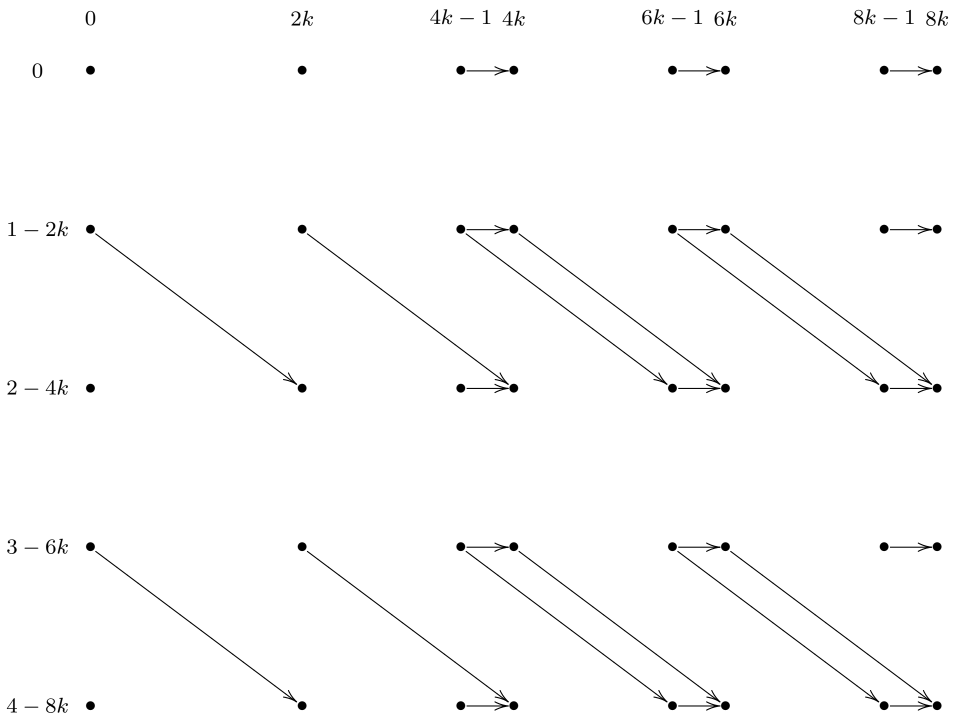
$$\partial(vz) = u^2 z$$

$$\partial(wz) = uz^2.$$

It follows that $\partial(uw - vz) = 0$. However, no other relations hold between the above five polynomials, for reasons of degree. We can illustrate this calculation by means of the spectral sequence of the double complex

$$\{F_C(w_{1-2k}, z_{2-4k}, u_{2k}, v_{4k-1}), d \otimes 1, 1 \otimes \delta\}.$$

The following diagram has dots wherever we have a generator at $E_0^{*,*}$ and arrows wherever we have a non-trivial d_n map.



Note that $(uw - vz)$ is not an element of $Im(\partial)$ so let $a := a_1 := uw - vz$. Note that $\partial(vw) = ua$ so that the ideal $(ua = 0)$ vanishes in homology.

However, $z^n a$ is in the kernel of ∂ but not in the image of ∂ (this again is for reasons of degree). Therefore, we conclude that

$$\mathbb{H}_*(L\mathbb{S}^{2k}) \cong \frac{F_C(u_{2k}, z_{2-4k}, a_1)}{(u_{2k}^2 = u_{2k}z_{2-4k} = u_{2k}a_1 = 0)}.$$

Note that the elements $u^i z^j$ for $i \geq 2$ get hit twice by boundary homomorphisms. This accounts for the element $a = uw - vz$ surviving to homology. We can also see (from the parallelograms in the above diagram) that vw hits both $u^2 w$ and uvz to kill ua (and hence kill every $u^i w^j a$ for $i, j \geq 1$).

2.5.3 Complex projective space.

An almost-free resolution of $S^*(\mathbb{C}P^n)$ is given by $M = (F_C(z_2, u_{2n+1}), \delta)$ where $\delta(u_{2n+1}) = z_2^{n+1}$ and $\delta(z_2) = 0$ (see for example [7]). Since $\mathbb{C}P^1$ is homeomorphic to $\mathbb{S}^2$, we already have the result of this calculation for $n = 1$. Indeed, the calculation of the general case is essentially no different to the even sphere calculation. Thus the computation is not given here, just the result

$$\mathbb{H}_*(L\mathbb{C}P^n) \cong \frac{F_C(u_2, z_{-2n}, a_1)}{(u_2^{n+1} = a_1 u_2^n = z_{-2n}u_2^n = 0)}.$$

2.5.4 The Grassmannian manifold.

Let $Gr_{n,k}$ denote the Grassmannian manifold of complex k-planes in $\mathbb{C}^n$. The cohomology ring of $Gr_{n,k}$ is given by

$$H^*(Gr_{n,k}) = \frac{F_C(z_2, z_4, \dots, z_{2k})}{(s_{2(n-k+1)} = \dots = s_{2n} = 0)}$$

where s_{2j} are the Sègre classes and are defined as polynomials in the z_{2i} by the formula

$$(1 + z_2 + \dots + z_{2k})(1 + s_2 + s_4 + \dots) = 1.$$

(This is well-known: see for example [26]). Therefore the almost free resolution of $S^*(Gr_{n,k})$ is given by

$$(F_C(z_2, \ldots, z_{2k}, u_{2(n-k+1)-1}, \ldots, u_{2n-1}), \delta)$$

where $\delta(u_{2j-1}) = s_{2j}$. Theorem 2.3.1 gives us an isomorphism

$$F_C(z_2, \ldots, z_{2k}, u_{2(n-k+1)-1}, \ldots, u_{2n-1})$$

$$\otimes F_C(w_{-1}, \ldots, w_{1-2k}, v_{2(k-n)}, \ldots, v_{2-2n}) \cong H^*(A, A).$$

Performing the same calculation as for complex projective space, we find that

$$\partial(z_i) = \partial(v_j) = 0,$$

$$\partial(u_{2i-1}) = s_{2i},$$

$$\partial(w_{2i-1}) = t_{2i},$$

where t_{2i} is a polynomial in the various z_i and v_j given as follows. For each polynomial s_{2j} and each z_{2i} we create polynomials $\bar{t}^{(2i)}_{2(j-i)}$ by replacing all instances of z^a_{2i} that occur in s_{2j} with az^{a-1}_{2i}. Then define

$$t_{-2l} = \sum_{i=1}^{k} v_{2(i-n)} \bar{t}^{(2l)}_{2(n-i-l)}.$$

It follows that

$$R := \frac{F_C(z_2, \ldots, z_{2k}, u_{2(n-k+1)-1}, \ldots, u_{2n-1}, w_{-1}, \ldots, w_{1-2k}, v_{2(k-n)}, \ldots, v_{2-2n})}{(s_{2(n-k+1)} = \cdots = s_{2n} = t_{-2k} = \cdots = t_0 = 0)}$$

is a subring of $\mathrm{HH}^*(A, A) \cong \mathbb{H}_*(LGr_{n,k})$. To check whether any other polynomials survive to homology or not, we calculate

$$\partial(u_{2i-1} v_{2(j-n)}) = s_{2i} v_{2(j-n)} \tag{2.3}$$

$$\partial(u_{2i-1}w_{1-2j}) = s_{2i}w_{1-2j} - u_{2i-1}t_{-2j} \tag{2.4}$$

$$\partial(v_{2(j-n)}w_{1-2j}) = v_{2(j-n)}t_{-2j} \tag{2.5}$$

$$\partial(w_{1-2j}z_{2i}) = t_{-2j}z_{2i} \tag{2.6}$$

$$\partial(z_{2i}u_{2i-1}) = z_{2i}s_{2i}. \tag{2.7}$$

No relations can hold between these for reasons of differing degree, except possibly when $i = j$, in which case there may be a relation between (2.3) and (2.6). However, due to the coefficients in t_{-2j}, no relations hold unless $k = 1$ (which gives the complex projective space case dealt with earlier). Thus for $k > 1$,

$$\mathbb{H}_*(LGr_{n,k}) \cong \frac{F_C(z_2, \ldots, z_{2k}, u_{2(n-k+1)-1}, \ldots, u_{2n-1}, w_{-1}, \ldots, w_{1-2k}, v_{2(k-n)}, \ldots, v_{2-2n})}{(s_{2(n-k+1)} = \cdots = s_{2n} = t_{-2k} = \cdots = t_0 = 0)}.$$

2.6 The Gerstenhaber bracket

In this section, we construct a Gerstenhaber algebra structure on $F_C(\Sigma V^*) \otimes F_C(V)$ so that the map described in theorem 2.3.1 is an isomorphism of Gerstenhaber algebras. The required Gerstenhaber bracket is given on generators as follows:

$$[f \otimes a, g \otimes b] = g(\alpha(a)) f \otimes b - (-1)^{(|f|+|a|)(|g|+|b|)} f(\alpha(b)) g \otimes b$$

for any $a, b \in V$ and $f, g \in \Sigma V^*$. For other elements of $F_C(\Sigma V^*) \otimes F_C(V)$, the formula for the Lie bracket can be found using the Gerstenhaber relation. The formulae above are used to compute some examples.

2.7.1 Example: Even dimensional spheres. As in section 2.5.2, let $u = u_{2k}$ be the generator of $\mathbb{H}_*(L\mathbf{S}^{2k})$ in degree $2k$; let $z = z_{2-4k}$ be the generator of $\mathbb{H}_*(L\mathbb{S}^{2k})$ in degree $2-4k$; let $a = a_1$ be the generator of $\mathbb{H}_*(L\mathbb{S}^{2k})$ in degree 1. Then

$$[u, z] = 0$$

$$[u, a] = u$$

$$[z, a] = z.$$

2.7.2 Example: Complex projective space. As in section 2.5.3, let $u = u_2$ be the generator of $\mathbb{H}_*(L\mathbb{C}P^n)$ in degree 2; let $z = z_{-2n}$ be the generator of $\mathbb{H}_*(L\mathbb{C}P^n)$ in degree $-2n$; let $a = a_1$ be the generator of $\mathbb{H}_*(L\mathbb{C}P^n)$ in degree 1. Then

$$[u, z] = 0$$

$$[u, a] = u$$

$$[z, a] = z.$$

Chapter 3

C_∞-algebras, L_∞-algebras and rational homotopy theory

3.1 Signs, grading and notation

In chapter 3, all the conventions of section 2.1 will be in place as well as the following:

For a graded vector space V, $F_L(V)$ is used to denote the free graded Lie algebra generated by V. $F^n_L(V)$ is used to denote the subspace generated by elements of the type $[\dots[a_1, a_2], a_3], \dots a_n]$ where $a_i \in V$ for each i. $F^n_C(V)$ is used to denote the subspace generated by elements of the type $a_1a_2 \dots a_n$ where $a_i \in V$ for each i.

For a graded K-vector space V with augmentation map $\epsilon : V \to K$, $\bar{V}$ is the kernel of ϵ. $\bar{F}_C(V)$ denotes the kernel of the augmentation map for $F_C(V)$. For a space X, $\bar{H}^*(X)$ denotes the reduced cohomology of the space.

Let K be a field of characteristic 0. K will be the ground field throughout this chapter. Particularly, for a space X, $\pi_*(X) = \pi_*(X) \otimes K$ and $\pi^*(X) = \mathrm{Hom}_K(\pi_*(X), K)$.

3.2 Basic definitions and concepts

The definitions of L_∞ and C_∞ algebras are not new (see e.g. [19] for L_∞-algebras and [24, section I.1.11] for C_∞-algebras) but are given here in order to introduce the notation required in defining L_∞- and C_∞-algebra cohomology. Let L be a differential graded Lie algebra with Lie bracket of degree zero and differential δ_1. The following commutative diagram defines a bracket on $\Sigma^{-1}L$ of degree 1. Both brackets will be denoted $[-,-]$.

$$\begin{array}{ccc} L \otimes L & \xrightarrow{[-,-]} & L \\ \downarrow{\scriptstyle \sigma^{-1}\otimes\sigma^{-1}} & & \downarrow{\scriptstyle \sigma^{-1}} \\ \Sigma^{-1}L \otimes \Sigma^{-1}L & \xrightarrow{[-,-]} & \Sigma^{-1}L \end{array}$$

This gives a co-Lie algebra structure on ΣL^*. Now define a map

$$\delta_2 : \Sigma L^* \to F_C(\Sigma L^*)$$

$$\delta_2\theta(X,Y) = (-1)^{|X|+1}\theta[X,Y]$$

wherever $\theta \in \Sigma L^*$ and $X, Y \in \Sigma^{-1}L$. The universal property of $F_C(-)$ allows us to extend δ_2 uniquely over $F_C(\Sigma L^*)$ to give a degree 1 map

$$\delta_2 : F_C(\Sigma L^*) \to F_C(\Sigma L^*)$$

that is a derivation of the commutative product on $F_C(\Sigma L^*)$. Let $\delta = \delta_1 + \delta_2$. The following can be shown by a direct calculation.

Lemma 3.2.1. *The derivation δ in $F_C(\Sigma L^*)$ is a differential. i.e. $\delta^2 = 0$.*

This gives rise to the following definitions (compare [24, section I.1.12], [16, section 4.3]):

Definition 3.2.1. Let V be a differential graded vector space with differential d. An L_∞*-algebra structure* on V is a degree 1 differential on $F_C(\Sigma V^*)$ that is a derivation of the product on $F_C(\Sigma V^*)$ and whose linear part is dual to d. If δ is such a differential, we say that (V, δ) is an L_∞*-algebra*.

Definition 3.2.2. A *co-*L_∞*-algebra structure* on V is a degree 1 differential on $F_C(\Sigma V)$ that is a derivation of the product on $F_C(\Sigma V)$ and whose linear part is dual to d. If δ is such a differential, we say that (V, δ) is a *co-*L_∞*-algebra*.

Definition 3.2.3. Let (V, δ) be an L_∞-algebra. The L_∞*-algebra cohomology* of V is the homology of $F_C(\Sigma V^*)$ with respect to δ. It is denoted $H^*(V; L_\infty)$.

Now let A be a differential graded commutative associative algebra with differential δ_1. Define

$$\delta_2 : F_L(\Sigma A^*) \to F_L(\Sigma A^*)$$

$$\delta_2 \theta(x, y) = (-1)^{|x|+1} \theta(xy)$$

for $\theta \in \Sigma A^*$, $x, y \in \Sigma^{-1} A$. Let $\delta = \delta_1 + \delta_2$. The following can be shown by a direct calculation.

Lemma 3.2.2. *The derivation δ in $F_L(\Sigma A^*)$ is a differential. i.e. $\delta^2 = 0$.*

This gives rise to the following definitions (compare [24, section I.1.12]):

Definition 3.2.4. Let V be a differential graded vector space with differential d. A C_∞*-algebra structure* on V is a degree 1 differential on $F_L(\Sigma V^*)$ that is a derivation of the product on $F_L(\Sigma V^*)$ and whose linear part is dual to d. If δ is such a differential, we say that (V, δ) is a C_∞*-algebra*.

Definition 3.2.5. A *co-C_∞-algebra structure* on V is a degree 1 differential on $F_L(\Sigma V)$ that is a derivation of the product on $F_L(\Sigma V)$ and whose linear part is dual to d. If δ is such a differential, we say that (V, δ) is a *co-C_∞-algebra.*

Note. C_∞-algebras are precisely commutative A_∞-algebras (see [24, section I.1.11]).

Definition 3.2.6. Let (V, δ) be a C_∞-algebra. The *C_∞-algebra cohomology* of V is the homology of $F_L(\Sigma V^*)$ with respect to δ. It is denoted $H^*(V; C_\infty)$.

3.3 Computing $\pi_*(X)$ from $H^*(X)$

Let X be a simply-connected CW-complex of finite type. In this section, we prove the following theorem:

Theorem 3.3.1. *There is a C_∞-structure on $\bar{H}^*(X)$ such that:*

(i) The linear part is zero.

(ii) The quadratic part is dual to the cup product.

(iii) There is an isomorphism of graded Lie algebras

$$H^*(\bar{H}^*(X); C_\infty) \cong \Sigma\pi_*(X).$$

Proof. Let A be a differential graded associative algebra. In [15, theorem 1], an A_∞-structure on $H_*(A)$ is constructed, together with a quasi-isomorphism of A_∞-algebras from $H_*(A)$ to A. Furthermore, this A_∞-structure on $H_*(A)$ has the properties required for parts (i) and (ii) of theorem 3.3.1.

To prove part (iii), we start by constructing a map of Lie algebras. Let A be a *minimal model* of $S^*(X)$: that is, an almost free resolution $(F_C(V), d)$ of $S^*(X)$ such that the the restricted map $d : V \to V$ is 0. Then by [15, theorem

1], we can construct a quasi-isomorphism of C_∞-algebras $\bar{H}^*(X) \to \bar{A}$. This induces an isomorphism of Lie algebras $H^*(\bar{A}; C_\infty) \xrightarrow{\cong} H^*(\bar{H}^*(X); C_\infty)$. Now, there is a map of differential graded vector spaces

$$f : \Sigma\bar{A}^* \to \Sigma\pi_*(X)$$

defined as follows. Let $V = \pi_*(X)$. Then [32] tell us that as a graded vector space $A = F_C(V^*)$. Since X is of finite type, V is finite dimensional. The inclusion $V^* \to \bar{A}$ induces a map $\bar{A}^* \to (V^*)^* \cong V$, the suspension of which is f. This extends uniquely to a map of graded Lie algebras

$$\theta : F_L(\Sigma\bar{A}^*) \to \Sigma\pi_*(X).$$

This in turn induces a map of graded Lie algebras $\theta_* : H^*(\bar{A}; C_\infty) \to \Sigma\pi_*(X)$. All that is required is to show that θ_* is an isomorphism.

The complex $F_L(\Sigma\bar{A}^*)$ can be regarded as a double complex

$$\{F_L^p(\Sigma A^*)_{p+q}, \delta_1, \delta_2\}$$

where $F_L^p(\Sigma\bar{A}^*)_{p+q}$ is the degree $p+q$ part of $F_L^p(\Sigma\bar{A}^*)$, δ_1 is the differential induced by the differential on A and δ_2 is the differential induced by the multiplication map $A \otimes A \to A$. By choosing δ_2 as the first differential, we obtain a spectral sequence. A simple calculation shows that the E^1 term of this spectral sequence is $\Sigma\pi_*(X)$ and that the sequence collapses at E^1. Hence the homology of $F_L(\Sigma\bar{A}^*)$ is $\Sigma\pi_*(X)$, so θ_* is an isomorphism. □

3.4 Computing $H^*(X)$ from $\pi_*(X)$

Theorem 3.4.1. *There is an L_∞-structure on $\Sigma\pi_*(X)$ such that:*

(i) The linear part is zero.

(ii) The quadratic part is dual to the Whitehead product.

(iii) There is an isomorphism of graded commutative algebras

$$H^*(\Sigma\pi_*(X); L_\infty) \cong H^*(X).$$

Proof. As mentioned previously, Sullivan showed in [32] that there is a differential operator δ on $F_C(\pi^*(X))$ such that $(F_C(\pi^*(X)), \delta)$ is quasi-isomorphic to $S^*(X)$. This means both that $\Sigma\pi_*(X)$ is an L_∞-algebra and that property (iii) holds. The differential δ gives rise to maps δ_n, defined as follows:

$$\delta_n : \pi^*(X) \xrightarrow{inc} F_C(\pi^*(X)) \xrightarrow{\delta} F_C(\pi^*(X)) \xrightarrow{proj} F_C^n(\pi^*(X)).$$

It is known (see for example [7, proposition 13.16]) that δ_2 is dual to the Whitehead product, giving property (ii). Property (i) arises from the fact that, when regarded as a differential graded vector space, $\Sigma\pi_*(X)$ has zero differential. □

We now give a geometric description of the L_∞-algebra structure on $\Sigma\pi_*(X)$. As mentioned in the previous proof, δ gives rise to various maps δ_n such that δ_2 is dual to the Whitehead product. In this chapter, higher Whitehead products are defined that are dual to δ_n for $n > 2$.

We can view the Whitehead product as the obstruction to extending $(f, g) : S^p \vee S^q \to X$ to a map $F : S^p \times S^q \to X$. In other words, let α, β be the homotopy classes of f, g respectively. The Whitehead product $[\alpha, \beta]$ is the homotopy class of $a \circ (f, g)$ where $a : S^{p+q-1} \to S^p \vee S^q$ is the attaching map so that $(S^p \vee S^q) \cup_a D^{p+q} = S^p \times S^q$. The product vanishes in $\pi_{p+q-1}(X)$ if and only if the extension F exists; that is, the following diagram commutes:

$$\begin{array}{ccc} S^p \times S^q & & \\ \uparrow & \searrow & \\ S^p \vee S^q & \to & X \end{array}$$

where the bottom arrow is (f,g) and the diagonal arrow is F.

Now suppose that $[\alpha,\beta]=0$. Then the extension to F exists. However, it is not unique. This is shown by looking at the cofibration sequence:

$$\cdots \to S^{p+q-1} \to S^p \vee S^q \longrightarrow S^p \times S^q \to S^{p+q} \to \dots$$

which induces an exact sequence:

$$\dots \longleftarrow \pi_{p+q-1}(X) \overset{\theta}{\longleftarrow} \pi_p(X) \oplus \pi_q(X) \overset{i}{\longleftarrow} [S^p \times S^q, X] \overset{\phi}{\longleftarrow} \pi_{p+q}(X) \leftarrow \dots.$$

If $[\alpha,\beta]=0$ then (α,β) is in the kernel of θ, so is in the image of i. Suppose $(\alpha,\beta)=i(\epsilon)$. Then $(\alpha,\beta)=i(\epsilon+\epsilon')$ for any ϵ' in the kernel of i. Therefore up to homotopy, there are as many extensions of (f,g) as there are elements in the kernel of i.

Suppose now that we have a map h representing $\gamma \in \pi_r(X)$ and that $[\beta,\gamma]=[\alpha,\gamma]=[\alpha,\beta]=0$. Let Y_m denote the m-skeleton of Y for any space Y and any positive integer m. Then the map (f,g,h) extends to a map $\phi:(S^p \times S^q \times S^r)_{p+q+r-1} \to X$. As discussed above, this extension is not unique. Suppose that there are k homotopy classes of such extensions. Let $\{\phi_1,\dots,\phi_k\}$ be a set of representatives of each of the k homotopy classes. For each i, we can ask whether ϕ_i extends to a map $S^p \times S^q \times S^r \to X$. Let

$$a : S^{p+q+r-1} \to (S^p \times S^q \times S^r)_{p+q+r-1}$$

be the attaching map so that $(S^p \times S^q \times S^r)_{p+q+r-1} \cup_a D^{p+q+r} \cong S^p \times S^q \times S^r$. Then we define the *Whitehead 3-product* of α,β,γ to be the set of homotopy classes $\{[\phi_1 \circ a],\dots,[\phi_k \circ a]\}$. It is denoted $[\alpha,\beta,\gamma]$. If any of $[\beta,\gamma]$, $[\alpha,\gamma]$, $[\alpha,\beta]$ are non-zero then define $[\alpha,\beta,\gamma]$ to be the empty set. If $[\alpha,\beta,\gamma]=\{0\}$ then we can see that each ϕ_i extends to a map $S^p \times S^q \times S^r \to X$. However, these extensions are not unique, by a similar argument to before.

Suppose inductively that the Whitehead $(n-1)$-product has been defined. Let $\alpha_1, \ldots, \alpha_n$ be elements of the homotopy groups $\pi_{p_1}(X), \ldots, \pi_{p_n}(X)$ respectively. Let $f_i : S^{p_i} \to X$ represent α_i for each i. Suppose that the Whitehead $(n-1)$-product of $\alpha_1, \ldots, \hat{\alpha}_i, \ldots \alpha_n$ is $\{0\}$ for each i. Then $(f_1, \ldots, f_n)$ extends to a set of maps $\phi_1, \ldots, \phi_k : (S^{p_1} \times \cdots \times S^{p_n})_{p_1+\cdots+p_n-1} \to X$. Let $a : S^{p_1+\cdots+p_n-1} \to (S^{p_1} \times \cdots \times S^{p_n})_{p_1+\cdots+p_n-1}$ be the attaching map so that $(S^{p_1} \times \cdots \times S^{p_n})_{p_1+\cdots+p_n-1} \cup_a D^{p_1+\cdots+p_n} \cong S^{p_1} \times \cdots \times S^{p_n}$. The *Whitehead n-product* of $\alpha_1, \ldots, \alpha_n$ is the set of homotopy classes $\{[\phi_1 \circ a], \ldots, [\phi_k \circ a]\}$. It is denoted by $[\alpha_1, \ldots, \alpha_n]$. If, for some i, the Whitehead $(n-1)$-product of $\alpha_1, \ldots, \hat{\alpha}_i, \ldots \alpha_n$ is not $\{0\}$ then define $[\alpha_1, \ldots, \alpha_n]$ to be the empty set.

Now we show that the Whitehead n-products arise naturally from the L_∞-structure on $\Sigma\pi_*(X)$. Firstly, define inner products

$$< -, - >: \pi^*(X) \otimes \pi_*(X) \to K$$

$$< v, \gamma >= v(\gamma)$$

and

$$< -; -, \ldots, - >: F^n_C(\pi^*(X)) \otimes \pi_*(X)^{\otimes n} \to K$$

$$< v_1 \ldots v_n; \gamma_1, \ldots, \gamma_n >= \sum_{\sigma \in S_n} \pm < v_1, \gamma_{\sigma(1)} > \cdots < v_n, \gamma_{\sigma(n)} >$$

for $v_1, \ldots, v_n \in \pi^*(X)$, where the sign in the sum is determined by the sign convention.

Theorem 3.4.2. *Let X be a simply connected space. Let $v \in \pi^*(X)$ and $\gamma_1, \ldots, \gamma_n \in \pi_*(X)$. Let $\gamma \in [\gamma_1, \ldots, \gamma_n]$. Then*

$$< \delta_n v; \gamma_1, \ldots, \gamma_n >= \pm < v, \gamma > .$$

Note. We can drop the 'simply connected' criterion from this theorem, but we then need to insist that v is not in $\mathrm{Hom}(\pi_1(X), K)$ and $\gamma_1, \dots, \gamma_n$ are not in $\pi_1(X) \otimes K$.

To prove 3.4.2, we use the following lemma (see [7, proposition 13.12] for a proof).

Lemma 3.4.3. *Let $\alpha \in \pi_n(X)$. Let $a : S^n \to X$ represent α. Let $(F_C(V), \delta)$ be an almost free resolution of $S^*(X)$. Let $V_+ = V \oplus Ku$ where u has degree $n+1$. Let δ_α be the derivation on $F_C(V_+)$ given by $\delta_\alpha u = 0$, $\delta_\alpha v = \delta v + < v, \alpha > u$ for every $v \in V$. Then $(F(V_+), \delta_\alpha)$ is an almost free resolution of $S^*(X \cup_a D^{n+1})$.*

We also need the following adjointness property, which follows directly from the definitions:

Lemma 3.4.4. *Let X, Y be topological spaces. Suppose that there is a map $f : X \to Y$. Let $f_* : \pi_*(X) \to \pi_*(Y)$ and $f^* : \pi^*(Y) \to \pi^*(X)$ be the maps induced by f. Then*

(i) $< f^*(a), b > = < a, f_*(b) >$

(ii) $< f^*(a); b_1, \dots, b_n > = < a; f_*(b_1), \dots, f_*(b_n) >$

for any $a \in \pi^*(Y)$, $b, b_1, \dots, b_n \in \pi_*(X)$.

Proof of theorem 3.4.2. First we assume that $X = (S^{p_1} \times \cdots \times S^{p_n})_{p_1 + \cdots + p_n - 1}$ where $p_1, \dots, p_n$ is a collection of integers, all of which are strictly greater than 1. Let e_i be image of the fundamental class of $H^{p_i}(S^{p_i})$ in q^*, where q is the projection $S^{p_1} \times \cdots \times S^{p_n} \to S^{p_i}$. Let $(F_C(V), d)$ be a minimal model for $S^*(S^{p_1} \times \cdots \times S^{p_n})$. Let x be an element of degree $p_1 + \cdots + p_n - 1$. Let W be the vector space spanned by x. Let δ be a differential on $F_C(V \oplus W)$ given by $\delta(x) = e_1 \dots e_n$ and $\delta(y) = d(y)$ for any $y \in V$. A simple calculation reveals the following isomorphism of algebras:

$$H^*(X) \cong \frac{H^*(S^{p_1} \times \cdots \times S^{p_n})}{(e_1 \dots e_n = 0)}.$$

Therefore $(F_C(V \oplus W), \delta)$ is a minimal model for $S^*(X)$. Let i_t be the element of $\pi_*(X)$ represented by the inclusion $S^{p_t} \hookrightarrow X$. Then

$$< \delta_n x; i_1, \dots, i_n >=< e_1 \dots e_n; i_1, \dots, i_n >= \pm 1$$

and for every $y \in V$,

$$< \delta_n y; i_1, \dots, i_n >= 0.$$

Let $\gamma \in [i_1, \dots, i_n]$. Then γ is the homotopy class of a map $a : S^{p_1+\cdots+p_n-1} \rightarrow X$ so we need to calculate $< x, \gamma >$. Using the notation of lemma 3.4.3, we have that

$$\delta_\gamma x = e_1 \dots e_n + < x, \gamma > u$$

where u is an element of degree $p_1 + \cdots + p_n$. Taking cohomology in $X \cup_a D^{p_1+\cdots+p_n}$, we have

$$[e_1 \dots e_n] + < x, \gamma > [u] = 0.$$

(Here, $[\alpha]$ means the cohomology class of α). Evaluating this at a generator of the homology group $H_{p_1+\cdots+p_n}(X \cup_a D^{p_1+\cdots+p_n})$ gives

$$< x, \gamma >= \pm 1$$

so that

$$< \delta_n x; i_1, \dots, i_n >= \pm < x, \gamma > .$$

For every $y \in V$, the degree of y is not equal to $-|\gamma|$, so $< y, \gamma >= 0$. It follows that

$$< \delta_n y; i_1, \dots, i_n >=< y, \gamma >$$

for every $y \in V \oplus W$.

Now suppose that X is an arbitrary (simply connected) space. Let $f_1, \ldots, f_n$ be maps representing $\gamma_1, \ldots, \gamma_n$ respectively. Suppose $f_t : S^{p_t} \to X$ for each t. Then the various f_t induce a map $\bar{F} : S^{p_1} \vee \cdots \vee S^{p_n} \to X$. If $\bar{F}$ does not extend to a map $(S^{p_1} \times \cdots \times S^{p_n})_{p_1+\cdots+p_n-1} \to X$ then the theorem is vacuously true since $[\gamma_1, \ldots, \gamma_n]$ is the empty set. So assume that there exists such an extension $F : (S^{p_1} \times \cdots \times S^{p_n})_{p_1+\cdots+p_n-1} \to X$. Let $\gamma \in [\gamma_1, \ldots, \gamma_t]$. Then $F_*(i_t) = \gamma_t$ so we have:

$$< \delta_n v; \gamma_1, \ldots, \gamma_n >=< \delta_n v; F_*(i_1), \ldots, F_*(i_n) > \tag{3.2}$$

and by lemma 3.4.4:

$$< \delta_n v; F_*(i_1), \ldots, F_*(i_n)) >=< F^*(\delta_n v); i_1, \ldots, i_n > . \tag{3.3}$$

Let i be the unique element of the Whitehead n-product $[i_1, \ldots, i_n]$ such that γ factors through i. Then, by the fact that the theorem is true for the space $(S^{p_1} \times \cdots \times S^{p_n})_{p_1+\cdots+p_n-1}$:

$$< F^*(\delta_n v); i_1, \ldots, i_n >= \pm < F^*(v), i > . \tag{3.4}$$

By lemma 3.4.4:

$$< F^*(v), i >=< v, F_*(i) > . \tag{3.5}$$

Since $F_*(i) = \gamma$, we have:

$$< v, F_*(i) >=< v, \gamma > \qquad (3.6).$$

The five equalities (3.2), (3.3), (3.4), (3.5), (3.6) show that $< \delta_n v; \gamma_1, \ldots, \gamma_n >= \pm < v, \gamma >$, proving the theorem. □

Note. The proof of this lemma is an adaption of the proof of [7, proposition 13.16], which deals with the quadratic part of the differential and the (ordinary) Whitehead product.

3.5 Formal and coformal spaces

It was noted in section 3.2 that there are C_∞-algebra structures on $H^*(X)$ and the almost free resolution A of $S^*(X)$ such that there is a quasi-isomorphism of C_∞-algebras

$$A \to H^*(X).$$

In [6], it was shown that for many interesting examples of X (for example, when X is a Kähler manifold), this quasi-isomorphism is also a quasi-isomorphism of ordinary commutative algebras (with the cup product). The spaces for which this is true are called *formal spaces*. We reinterpret this definition in terms of the *commutative algebra cohomology* of a commutative algebra.

Let V be a differential graded commutative algebra with product map $\tilde{d}_2$ and differential $\tilde{d}_1$. Then V^* is a cocommutative algebra with coproduct map $\tilde{\delta}_2$ induced from $\tilde{d}_2$ and differential $\tilde{\delta}_1$ induced from $\tilde{d}_1$. Let $\delta_n = inc \circ \sigma^{\otimes n} \circ \tilde{\delta}_n \circ \sigma^{-1} : \Sigma V^* \to F_L^n(\Sigma V^*) \hookrightarrow F_L(\Sigma V^*)$ for $n = 1, 2$ and let $\delta := \delta_1 + \delta_2$.

Definition 3.5.1. The *commutative algebra cohomology* of V is the homology of $(F_L(\Sigma V^*), \delta)$. It is denoted $H^*(V; C)$.

It now follows that a simply connected CW-complex X is formal if and only if

$$H^*(H^*(X); C) \cong H^*(H^*(X); C_\infty).$$

The purpose of this rephrasing of formality for CW-complexes is that we can dualise it to get a natural definition of a *coformal CW-complex*. That is, let V be a differential graded Lie algebra with bracket $\tilde{d}_2$ and differential $\tilde{d}_1$. Then V^* is a co-Lie algebra with coproduct map $\tilde{\delta}_2$ induced from $\tilde{d}_2$

and differential $\tilde{\delta}_1$ induced from $\tilde{d}_1$. Let $\delta_n = inc \circ \sigma^{\otimes n} \circ \tilde{\delta}_n \circ \sigma^{-1} : \Sigma V^* \to F_C^n(\Sigma V^*) \hookrightarrow F_C(\Sigma V^*)$ for $n = 1, 2$ and let $\delta := \delta_1 + \delta_2$.

Definition 3.5.2. The *Lie algebra cohomology* of ΣV is the homology of $(F_C(V^*), \delta)$. It is denoted $H^*(V; L)$.

Definition 3.5.3. A CW-complex X is *coformal* if and only if

$$H^*(\Sigma\pi_*(X); L) \cong H^*(\Sigma\pi_*(X); L_\infty).$$

Examples of coformal spaces are spheres and Stiefel manifolds. Also, the based loop space of a simply connected CW-complex is coformal (see e.g. [7]). However, complex projective space $\mathbb{C}P^n$, for $n > 1$, is not a coformal space since the Whitehead $(n+1)$-bracket is nontrivial. Indeed, $H^*(\Sigma\pi_*(\mathbb{C}P^n); L) \cong F_C(z_2, u_{2n+1})$ but $H^*(\Sigma\pi_*(\mathbb{C}P^n); L_\infty) \cong \frac{F_C(z_2)}{(z^{n+1}=0)}$.

Chapter 4

The Γ-homology of characteristic p cochains

Let p be a prime number. Let $\mathbb{F}_p$ be a field of characteristic p. Throughout this chapter, we will assume that $\mathbb{F}_p$ is the ground field. Let $S^*(X)$ denote the cochains on a p-complete nilpotent space X of finite p-type with coefficients in $\mathbb{F}_p$. Let $\mathcal{E}$ be an E_∞-operad over $\mathbb{F}_p$ that acts on $S^*(X)$. Let $\Gamma H_*(A)$ denote the Robinson-Whitehead Γ-homology of A (see [28]). The main theorem of this chapter is

Theorem 4.0.1. $\Gamma H_*(S^*(X)) \cong 0$.

4.1 The Γ-homology of $\mathcal{E}$-algebras

Let A be an $\mathcal{E}$-algebra. The Γ-homology of A was defined in [28]. For convenience and to introduce notation, the definition is given here. Let U_A be the quotient of

$$\bigoplus_{k\geq 2} \mathcal{E}(k) \otimes_{\Sigma_k} A^{\otimes k}$$

by the following identifications.

Let $k, k_1, k_2 \in \mathbb{Z}_{\geq 1}$ be such that $k_1 + k_2 - 1 = k$. Let

$$\chi : \mathcal{E}(k_2) \otimes \mathcal{E}(k_1) \otimes \mathcal{E}(1)^{\otimes k_2 - 1} \to \mathcal{E}(k)$$

be the structure map making $\mathcal{E}$ an operad. Let

$$\partial^{k_1,k_2} : \mathcal{E}(k_1) \otimes \mathcal{E}(k_2) \otimes A^{\otimes k} \to \mathcal{E}(k_1) \otimes A^{\otimes k_1} \oplus \mathcal{E}(k_2) \otimes A^{\otimes k_2}$$

be $\partial^{k_1,k_2} = (1 \otimes \rho \otimes 1^{\otimes k_1 - 1}) \circ (1 \otimes 1 \otimes t) \oplus (1 \otimes \rho \otimes 1^{\otimes k_2 - 1})(\tau \otimes 1^{\otimes k})$ where ρ is the structure map making A an $\mathcal{E}$-algebra, t interchanges $A^{\otimes k_1} \otimes A^{\otimes k_2}$ with $A^{\otimes k_2} \otimes A^{\otimes k_1}$ and τ interchanges $\mathcal{E}(k_1)$ and $\mathcal{E}(k_2)$. The identification we wish to make is

$$\chi(x \otimes y \otimes z) \otimes m \sim \partial^{k_1,k_2}(y \otimes x \otimes m)$$

where $x \in \mathcal{E}(k_2)$, $y \in \mathcal{E}(k_1)$, $z \in \mathcal{E}(1)^{\otimes k_2}$ is the identity and $m \in A^{\otimes k}$. The reason for this identification is that the structure map sends both elements $\chi(x \otimes y \otimes z) \otimes m$ and $\partial^{k_1,k_2}(y \otimes x \otimes m)$ of $\bigoplus_{k \geq 2} \mathcal{E}(k) \otimes_{\Sigma_k} A^{\otimes k}$ to the same element in A. Therefore (as was shown in [28, section 2.8]) the structure map

$$\bigoplus_{k \geq 2} \mathcal{E}(k) \otimes_{\Sigma_k} A^{\otimes k} \to A$$

factors through U_A to give a map

$$\theta_A : U_A \to A.$$

Notation. If $x \in \bigoplus_{k \geq 2} \mathcal{E}(k) \otimes_{\Sigma_k} A^{\otimes k}$ then we denote by $[x]$ the corresponding equivalence class in U_A.

For a chain complex V, let CV be the cone on V. That is, if δ is the differential on V then CV is the chain complex whose underlying vector space is $V \oplus \Sigma^{-1}V$ and whose boundary map is $\delta + d$, where d is induced by the suspension map $\Sigma^{-1}V \to V$. Let A_Γ be the *cofiber* of θ_A: that is, the unique chain complex such that the following is a pushout square

$$\begin{array}{ccc} U_A & \xrightarrow{i_A} & CU_A \\ {\scriptstyle\theta_A}\downarrow & & \downarrow \\ A & \longrightarrow & A_\Gamma \end{array}.$$

The map $A \mapsto A_\Gamma$ is the 'appropriate indecomposables functor' mentioned in section 1.3.

Definition 4.1.1. The Γ-*homology of* A is defined to be the homology of A_Γ. It is denoted $\Gamma H_*(A)$.

The following property of A_Γ will be used several times.

Proposition 4.1.1. *Let*

$$\begin{array}{ccc} A & \xrightarrow{\beta} & B \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle b} \\ C & \underset{c}{\longrightarrow} & D \end{array}$$

be a pushout diagram. Let E be the pushout of $C_\Gamma \stackrel{\gamma_\Gamma}{\leftarrow} A_\Gamma \stackrel{\beta_\Gamma}{\rightarrow} B_\Gamma$. Suppose at least one of γ_Γ and β_Γ are injective. Then the canonical map $E \to D_\Gamma$ is a quasi-isomorphism.

In order to prove this, we will need the following lemma.

Lemma 4.1.2. *If $X \stackrel{f}{\rightarrow} Y \stackrel{g}{\rightarrow} Z \to 0$ is exact then $X_\Gamma \stackrel{f_\Gamma}{\rightarrow} Y_\Gamma \stackrel{g_\Gamma}{\rightarrow} Z_\Gamma \to 0$ is exact.*

Proof. Let A be a $\mathcal{E}$-algebra. Since $i_A : U_A \to CU_A$ is injective,

$$0 \to U_A \stackrel{(\theta_A, i_a)}{\rightarrow} A \oplus CU_A \to A_\Gamma \to 0$$

is exact so that

$$A_\Gamma \cong \frac{A \oplus CU_A}{(\theta_A, i_A)(U_A)}.$$

Since $X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is exact, it follows that

$$X \oplus CU_X \xrightarrow{\tilde{f}} Y \oplus CU_Y \xrightarrow{\tilde{g}} Z \oplus CU_Z \to 0$$

is exact, where $\tilde{f}$ and $\tilde{g}$ are induced from f and g respectively. Now let $y + Im(\theta_Y, i_Y)$ be an element of $Ker(g_\Gamma)$. Then $\tilde{g}(y)$ is an element of $Im(\theta_Z, i_Z)$. We construct some $y' \in Im(\theta_Y, i_Y)$ such that $g(y) = g(y')$. This is done as follows. Write $\tilde{g}(y)$ in the form

$$([\phi_Z^{(n)}(e^{(0)} \otimes (e^{(1)} \otimes z_1^{(1)} \otimes z_2^{(1)}) \otimes \cdots \otimes (e^{(k)} \otimes z_1^{(k)} \otimes z_2^{(k)}))], \bar{g}(\alpha)) = (\theta_Z, i_Z)(\bar{g}(\alpha))$$

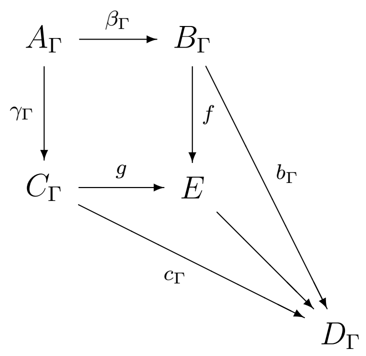
where $\bar{g} : CU_Y \to CU_Z$ is the map induced by g; α is an element of $U_Y^{(n)}$ or, ambiguously, its image in i_Y; $z_j^{(i)} \in Z$ and $e^{(i)} \in \mathcal{E}$. Since $\tilde{g}$ is surjective, there exists $y_j^{(i)} \in Y$ such that $\tilde{g}(y_j^{(i)}) = z_j^{(i)}$ for each i, j. Let

$$y' = ([\phi_Y^{(n)}(e^{(0)} \otimes (e^{(1)} \otimes y_1^{(1)} \otimes y_2^{(1)}) \otimes \cdots \otimes (e^{(k)} \otimes y_1^{(k)} \otimes y_2^{(k)}))], \alpha).$$

Then $\tilde{g}(y) = \tilde{g}(y')$ as required, so that $y - y' \in Ker(\tilde{g}) = im(\tilde{f})$. Let $x \in X$ be such that $\tilde{f}(x) = y - y'$. Then $f_\Gamma(x + Im(\theta_X, i_X)) = \tilde{f}(x) + Im(\theta_Y, i_Y) = y - y' + Im(\theta_Y, i_Y) = y + Im(\theta_Y, i_Y)$ so that $y + Im(\theta_Y, i_Y) \in Im(f_\Gamma)$. Thus $Ker(g_\Gamma) \subseteq Im(f_\Gamma)$.

The proofs of the facts that $Im(\theta_Y, i_Y) \subseteq Ker(g_\Gamma)$ and g_Γ is surjective are simple, direct checks. □

Proof of proposition 4.1.1. We have the following commutative diagram:



Since E is a pushout,

$$A_\Gamma \overset{(\beta_\Gamma,\gamma_\Gamma)}{\to} B_\Gamma \oplus C_\Gamma \overset{f-g}{\to} E \to 0$$

is exact. Similarly,

$$A \overset{(\beta,\gamma)}{\to} B \oplus C \overset{b-c}{\to} D \to 0$$

is exact so, by lemma 4.1.2,

$$A_\Gamma \overset{(\beta_\Gamma,\gamma_\Gamma)}{\to} B_\Gamma \oplus C_\Gamma \overset{b_\Gamma - c_\Gamma}{\to} D_\Gamma \to 0$$

is exact. Since at least one of β_Γ and γ_Γ are injective, $(\beta_\Gamma, \gamma_\Gamma)$ is injective so that

$$\begin{array}{ccccccccc} 0 & \to & A_\Gamma & \overset{(\beta_\Gamma,\gamma_\Gamma)}{\to} & B_\Gamma \oplus C_\Gamma & \overset{f-g}{\to} & E & \to & 0 \\ & & \downarrow = & & \downarrow = & & \downarrow h & & \\ 0 & \to & A_\Gamma & \overset{(\beta_\Gamma,\gamma_\Gamma)}{\to} & B_\Gamma \oplus C_\Gamma & \overset{b_\Gamma - c_\Gamma}{\to} & D_\Gamma & \to & 0 \end{array}$$

commutes and the rows are exact. The result then follows from the 5-lemma.

□

4.2 A proof of theorem 4.0.1

Let $K(\mathbb{Z}_p, n)$ be an Eilenberg-Maclane space whose n-th homotopy group is $\mathbb{Z}_p$ (that is, the cyclic group with p elements). In [20, section 6], a cofibrant resolution of $S^*(K(\mathbb{Z}_p,n))$ is constructed in the category of $\mathcal{E}$-algebras. We sketch the construction here.[1] Let $\mathbb{F}_p[n]$ be the graded vector space over $\mathbb{F}_p$ of dimension one, generated by one element x_n in degree n. Now, the homology of any E_∞-algebra is an algebra over the generalised Steenrod algebra. (See [23] for details on the generalised Steenrod algebra. See also [17, chapter 1] for a more modern treatment.) Let P^0 be the degree 0 generator of the generalised Steenrod algebra. Let $\bar{p}^0 : \mathbb{F}_p[n] \to F_{\mathcal{E}}(\mathbb{F}_p[n])$ be the map taking the generator of $\mathbb{F}_p[n]$ to a representative of the homology class P^0x_n. Let $p^0 : F_{\mathcal{E}}(\mathbb{F}_p[n]) \to F_{\mathcal{E}}(\mathbb{F}_p[n])$ denote the induced map of $\mathcal{E}$-algebras. Let $\bar{a} : \mathbb{F}_p[n] \to S^*(K(\mathbb{Z}_p,n);\mathbb{F}_p)$ be the map taking the generator of $\mathbb{F}_p[n]$ to a representative of the fundamental class of $H^n(K(\mathbb{Z}_p,n);\mathbb{F}_p)$ and $a : F_{\mathcal{E}}(\mathbb{F}_p[n]) \to S^*(K(\mathbb{Z}_p,n);\mathbb{F}_p)$ be the induced map of $\mathcal{E}$-algebras. Let $\bar{i} : \mathbb{F}_p[n] \to C\mathbb{F}_p[n]$ be the inclusion and $i : F_{\mathcal{E}}(\mathbb{F}_p[n]) \to F_{\mathcal{E}}(C\mathbb{F}_p[n])$ the induced map of $\mathcal{E}$-algebras. Let α be the fundamental class of $H^n(K(\mathbb{Z}_p,n))$. Let $x \in S^n(K(\mathbb{Z}_p,n))$ be a representative of α. Let $x^0 \in S^n(K(\mathbb{Z}_p,n))$ be a representative of $P^0\alpha$. Let $q_{n-1} \in S^{n-1}(K(\mathbb{Z}_p,n))$ be an element such that $dq_{n-1} = x - x^0$ (such an element exists since $1-P^0$ is zero on $H^*(K(\mathbb{Z}_p,n);\mathbb{F}_p)$ so $x - x^0$ is a boundary). Let $\bar{q} : C\mathbb{F}_p[n] \to S^*(K(\mathbb{Z}_p,n);\mathbb{F}_p)$ take the degree $n-1$ generator to q_{n-1} and the degree n generator to $x - x^0$. Let $q : F_{\mathcal{E}}(C\mathbb{F}_p[n]) \to S^*(K(\mathbb{Z}_p,n);\mathbb{F}_p)$ be the induced map of $\mathcal{E}$-algebras. Then the following diagram commutes:

$$\begin{array}{ccc} F_{\mathcal{E}}(\mathbb{F}_p[n]) & \xrightarrow{i} & F_{\mathcal{E}}(C\mathbb{F}_p[n]) \\ {\scriptstyle 1-p^0}\downarrow & & \downarrow{\scriptstyle q} \\ F_{\mathcal{E}}(\mathbb{F}_p[n]) & \xrightarrow{a} & S^*(K(\mathbb{Z}_p,n);\mathbb{F}_p) \end{array}.$$

Furthermore, the pushout, B, of $F_{\mathcal{E}}(\mathbb{F}_p[n]) \xleftarrow{1-p^0} F_{\mathcal{E}}(\mathbb{F}_p[n]) \xrightarrow{i} F_{\mathcal{E}}(C\mathbb{F}_p[n])$ is

[1] In fact, in section 6 of [20], this construction is given when the ground field is the algebraic closure of the field $\mathbb{F}_p$, but in the proof of [20, proposition A.7], Mandell notes that the construction is just as valid for any field of characteristic p.

$\mathcal{E}$-algebra cofibrant and the canonical $\mathcal{E}$-algebra map $B \to S^*(K(\mathbb{Z}_p, n); \mathbb{F}_p)$ is a quasi-isomorphism (this was shown in [20]).

Now, [28, proposition 4.1] says that $F_{\mathcal{E}}(V)_\Gamma \cong V$ for any differential graded vector space V. Therefore, by applying the functor $A \mapsto A_\Gamma$ to the pushout diagram

$$\begin{array}{ccc} F_{\mathcal{E}}(\mathbb{F}_p[n]) & \xrightarrow{i} & F_{\mathcal{E}}(C\mathbb{F}_p[n]) \\ {\scriptstyle 1-p^0}\downarrow & & \downarrow \\ F_{\mathcal{E}}(\mathbb{F}_p[n]) & \longrightarrow & B \end{array}$$

we get the following commutative diagram:

$$\begin{array}{ccc} \mathbb{F}_p[n] & \xrightarrow{i_\Gamma} & C\mathbb{F}_p[n] \\ {\scriptstyle 1-p^0_\Gamma}\downarrow & & \downarrow \\ \mathbb{F}_p[n] & \longrightarrow & B_\Gamma \end{array}.$$

Let D be the pushout of $\mathbb{F}_p[n] \xleftarrow{1-p^0_\Gamma} \mathbb{F}_p[n] \xrightarrow{i_\Gamma} C\mathbb{F}_p[n]$. Then we have a short exact sequence

$$0 \to \mathbb{F}[n] \xrightarrow{(1-p^0_\Gamma, inc)} \mathbb{F}_p[n] \oplus C\mathbb{F}_p[n] \to D \to 0$$

giving a long exact sequence in homology, the non-trivial part of which is

$$0 \to H_{n-1}(D) \to \mathbb{F}_p \xrightarrow{(1-p^0_\Gamma)_*} \mathbb{F}_p \to H_n(D) \to 0.$$

By lemma 4.1.2, $H_n(D) \cong H_n(B_\Gamma)$ so the following sequence is exact:

$$0 \to H_{n-1}(B_\Gamma) \to \mathbb{F}_p \xrightarrow{(1-p^0_\Gamma)_*} \mathbb{F}_p \to H_n(B_\Gamma) \to 0.$$

The result for $X = K(\mathbb{Z}_p, n)$ follows from the following lemma.

Lemma 4.2.1. *p^0_Γ is homologous to the zero map.*

Proof. p^0 takes x_n to an element in the image of θ_A. Thus $p^0_\Gamma(x_n)$ is null-homologous. It follows that the map p^0_Γ is homologous to the zero map. □

Lemma 4.2.2. *Let $m \in \mathbb{Z}_{>0}$. Then $\Gamma H_*(S^*(K(\mathbb{Z}_{p^m}, n))) \cong 0$.*

Proof. Again, we use a result of [20, section 7] to obtain an $\mathcal{E}$-algebra cofibrant model for $S^*(K(\mathbb{Z}_{p^m}, n))$. We sketch the construction here. Let $m > 1$. Inductively, suppose that a cofibrant model, $B^{m-1,n}$, for $S^*(K(\mathbb{Z}_{p^{m-1}}, n))$ has been constructed for each n. Let $\beta_m : B^{1,n+1} \to B^{m-1,n}$ be a map constructed by taking a representative of the fundamental class of $H_*(B^{1,n+1})$ to a representative of the m-th Bockstein class of $H_*(B^{m-1,n})$. Let $CB^{1,n+1}$ be the cone on $B^{1,n+1}$. Define $B^{m,n}$ by the pushout diagram

$$\begin{array}{ccc} B^{1,n+1} & \longrightarrow & CB^{1,n+1} \\ {\scriptstyle \beta_m}\downarrow & & \downarrow \\ B^{m-1,n} & \longrightarrow & B^{m,n} \end{array}.$$

Since the top arrow is injective, by proposition 4.1.1 we have a short exact sequence

$$0 \to B_\Gamma^{1,n+1} \to B_\Gamma^{m-1,n} \oplus CB_\Gamma^{1,n+1} \to D \to 0$$

where $H_n(D) \cong H_n(B_\Gamma^{m,n})$. The resulting long exact sequence in homology shows that if $H_*(B_\Gamma^{m-1,n}) = 0$ for every n then $H_*(B_\Gamma^{m,n}) = 0$ for every n so lemma 4.2.2 is proved by induction on m. □

Corollary 4.2.3. *Let $\mathbb{Z}_{p^\infty}$ denote the p-adic integers $colim_m(\mathbb{Z}_{p^m})$. Then $\Gamma H_*(S^*(K(\mathbb{Z}_{p^\infty}))) \cong 0$.*

Proof of theorem 4.0.1. Let X be a p-complete nilpotent space of finite p-type. Then X has a principally refined Postnikov tower whose fibers are of the type $K(\mathbb{Z}_p, n)$ or $K(\mathbb{Z}_{p^\infty}, n)$. Let $\cdots \to X_{n+1} \to X_n \to \cdots \to X_0$ be the Postnikov tower of X (so that the canonical map $X \to X_n$ induces an isomorphism on m-th homotopy groups for $m \leq n$ and $\pi_m(X_n) = 0$ for $m > n$). Then (as is standard - see e.g. [12, p. 410ff.]) we have a commutative diagram:

$$\begin{array}{ccc} X_{n+1} & \longrightarrow & PK(\pi_{n+1}(X), n+2) \\ \downarrow & & \downarrow \\ X_n & \longrightarrow & K(\pi_{n+1}(X), n+2) \end{array} .$$

X_1 is an Eilenberg-Maclane space by definition, so suppose inductively that there is an isomorphism $0 \to \Gamma H_*(A^n)$ where A^n is a cofibrant resolution of $S^*(X_n)$. Now, since $\pi_{n+1}(X)$ is either $\mathbb{Z}_p$ or $\mathbb{Z}_{p^\infty}$, the above diagram gives a pushout square

$$\begin{array}{ccc} B^{n+2,m} & \longrightarrow & CB^{n+2,m} \\ \downarrow & & \downarrow \\ A^n & \longrightarrow & A^{n+1} \end{array}$$

where $m \in \{1, \infty\}$, $B^{k,m}$ is an $\mathcal{E}$-algebra cofibrant resolution of $S^*(K(\mathbb{Z}_{p^m}, k))$ and A^{n+1} is an $\mathcal{E}$-algebra cofibrant resolution of $S^*(X_{n+1})$. Applying the functor $A \mapsto A_\Gamma$ gives a pushout diagram

$$\begin{array}{ccc} B_\Gamma^{n+2,m} & \longrightarrow & CB_\Gamma^{n+2,m} \\ \downarrow & & \downarrow \\ A_\Gamma^n & \longrightarrow & A_\Gamma^{n+1} \end{array} .$$

Since the map $B_\Gamma^{n+2,m} \to CB_\Gamma^{n+2,m}$ is an injection, we have a short exact sequence

$$0 \to B_\Gamma^{n+2,m} \to A_\Gamma^n \oplus CB_\Gamma^{n+2,m} \to A_\Gamma^{n+1} \to 0$$

which gives a long exact sequence

$$\cdots \to H_k(B_\Gamma^{n+2,m}) \to H_k(A_\Gamma^n) \to H_k(A_\Gamma^{n+1}) \to H_{k+1}(B_\Gamma^{n+2,m}) \to \ldots .$$

Since $H_k(B_\Gamma^{n+2,m}) = 0$ and $H_{k+1}(B_\Gamma^{n+2,m}) = 0$, it follows that $H_k(A_\Gamma^n) \cong H_k(A_\Gamma^{n+1})$. Now, $H_k(A_\Gamma^1) = 0$ since A_1 is a quasi-isomorphic to the cochains on an Eilenberg-Maclane space. Therefore $H_k(A_\Gamma^n) = 0$ for each k, n. Hence the Γ-homology of $S^*(X_n)$ is zero so the Γ-homology of $S^*(X)$ is zero. □

Chapter 5

The step operad with $\mathbb{Z}_2$ coefficients

In this chapter we construct an operad $\mathcal{S}$ over $\mathbb{Z}_2$, called the *step operad* and show that for any topological space X, $S_*(X;\mathbb{Z}_2)$ is a coalgebra over $\mathcal{S}$. Since we work with singular chains on a space in this chapter, all differentials will decrease degree by 1. This convention will hold for the rest of the thesis. We begin by introducing the concept of (pre-)step diagrams.

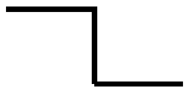
5.1 Step diagrams

Before giving a formal definition of step diagrams, it is motivational to look at the idea behind their construction. This comes from an attempt to form an operad from Steenrod's cup-i products (and their iterates). In [30], the cup-i products are constructed via the concept of i-regularity. Let Δ^N be an N-simplex with vertices $v_0, \ldots, v_N$. Let σ, τ be sub-simplices of Δ^N. If the vertices of σ are $v_0, \ldots, v_i$ and the vertices of τ are $v_i, \ldots, v_N$ then Steenrod said that (σ, τ) is a 0-*regular pair of sub-simplices.* We can draw a diagram of 0-regularity as follows:

$\sigma \quad v_0, \ldots, v_i$

$\tau \qquad\qquad v_i, \ldots, v_N$

In other words, when written like this, the vertices of a 0-regular pair of simplices form a shape that looks like:



This is the first example of a *step diagram*.

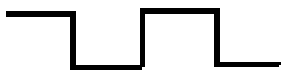
Now suppose the vertices of σ are $v_0, \ldots, v_i, v_j, \ldots, v_N$ and the vertices of τ are $v_i, \ldots, v_j$ $(j \neq i)$. Then (σ, τ) is said to be a 1-*regular pair of sub-simplices*. We can draw a diagram of 1-regularity as follows:

$\sigma \quad v_0, \ldots, v_i \qquad v_j, \ldots, v_N$

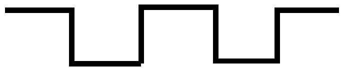
$\tau \qquad\qquad v_i, \ldots, v_j$

In other words, when written like this, the vertices of a 1-regular pair of simplices form a shape that looks like:

This is another example of a step diagram. Continuing in this manner, the vertices of 2-regular pair of simplices form a shape that looks like the step diagram:



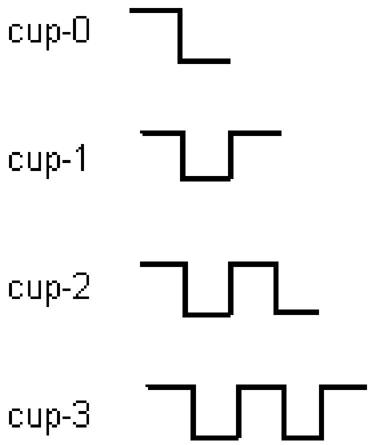
A 3-regular pair of simplices forms a shape that looks like the step diagram:



...and so on. In [30], Steenrod used i-regularity to define the cup-i product $a \cup_i b$ of two cochains a, b as follows. Let a, b be cochains dual to singular simplices σ and τ respectively. Let ϕ be a singular n-simplex. If σ and τ are restrictions of ϕ to faces f_1, f_2 of Δ^n such that f_1, f_2 is an i-regular pair of subsimplices of Δ^n, then we say that σ, τ is an *i-regular pair of subsimplices of ϕ*. Steenrod defines

$$a \cup_i b(\phi) = \begin{cases} 1 \text{ if } \sigma, \tau \text{ is an } i\text{-regular pair of subsimplices of } \phi \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise} \end{cases}$$

In fact, [30] describes products on the *simplicial* cochain complex of a simplicial complex. However, Steenrod's arguments work equally well in the case of the singular cochain complex of a topological space. Therefore we can think of each of the above step diagrams as defining a product on the cochain complex of a topological space; or equally, a coproduct on the chain complex. This suggests the following code for the cup-i products:



...and so on. Now let us look at the step diagrams that correspond to iterated cup-i products (for example $(x \cup_i y) \cup_j z)$. The step diagrams will look something like one of the four diagrams on the next page. The heights correspond to the number of cup products used (for example a linear combination of step diagrams of height 2 would correspond to $(x \cup_i y) \cup_j z)$.

These four step diagrams are just randomly picked examples. The general idea is that a step diagram is drawn by starting on the left with a horizontal line segment, then alternating between vertical and horizontal line segments (all connected together) and ending with a horizontal line segment. The horizontal line segments are called *steps*. Each of the vertical line segments are of integer length, so the steps can be assigned a natural *height*, with the lowest step being, by convention, height 1. If the highest step in the diagram is height n then we insist that there is a step in the diagram of height i for each $1 \leq i \leq n$. There are several ways that this idea can be formalised. The following formalisation will be used for the rest of this thesis, except where otherwise stated.

Definition 5.1.1. A *pre-step diagram* is a 1-dimensional simplicial complex S with the following properties:

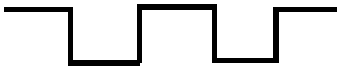
(i) S has $2m$ vertices $v_1, \ldots, v_{2m}$ for some positive integer m and $2m-1$ edges $e_1, \ldots, e_{2m-1}$ such that the boundary of e_i is $v_i \cup v_{i+1}$.

(ii) The edges come with an ordering: $e_i < e_j$ if and only if $i < j$.

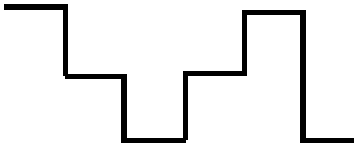
(iii) The edges e_{2i} are called *vertical edges* and the edges e_{2i-1} are called *horizontal edges* or *steps*.

(iv) Each vertical edge e_{2i} has an non-zero integer associated to it, called the *length* of e_{2i} and denoted $l(e_{2i})$.

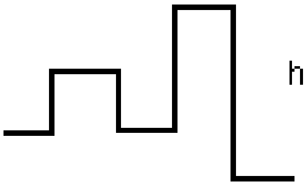
Definition 5.1.2. Let S be a pre-step diagram with edges $e_1, \ldots, e_{2m-1}$. Let $i \in \{1, \ldots, m\}$. Let $h'(e_{2i-1}) = \sum_{j<i} l(e_{2j})$. Define the *height* of edge e_{2i-1}



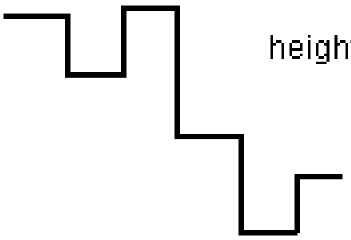
height 1



height 2



height 3



height 4

to be

$$h(e_{2i-1}) = h'(e_{2i-1}) - min\{h'(e_{2i-1})|j \in \{1, \ldots, m\}\} + 1.$$

Definition 5.1.3. Let S be a pre-step diagram. The *height* of S is defined to be

$$h(S) = max\{h_i|i \in \{1, \ldots, m\}\} - 1.$$

Notice that a pre-step diagram S is uniquely determined by specifying the number of steps in S, the order of the steps and the height of each step. If S is a pre-step diagram with edges $e_1, \ldots, e_{2m-1}$ then we denote

$$S =< h(e_1), h(e_3), \ldots, h(e_{2m-1}) > .$$

A pre-step diagram can be uniquely specified in this way. By using this idea, it is possible to formalise the notion of a pre-step diagram in a different way: by saying that it is a finite ordered sequence of positive integers with the property that 1 must occur somewhere in the sequence. The idea of defining pre-step diagrams like this is used in section 6.3. For now, we will stick to the simplicial complex definition (definitions 5.1.1 and 5.1.2).

Definition 5.1.4. A *step diagram* of height $k-1$ is a pre-step diagram $S =< h_1, \ldots, h_m >$ of height $k-1$ with the property that for every $s \in \{1, \ldots, k\}$, there is some $j \in \{1, \ldots, m\}$ such that $h_j = s$.

Examples. Figure 1 (overpage) is a drawing of $S =< 3, 2, 1, 3, 2 >$. The pre-step diagram $< 1, 2, 3, 5, 3, 2, 3 >$ in figure 2 is not a step diagram, since it includes a step of height 5 but has no step of height 4.

5.2 Deleting a step

Definition 5.2.1. Let $S =< h_1, \ldots, h_m >$ be a step diagram with edges $e_1, \ldots, e_{2m-1}$. For $i \in \{1, \ldots, m, \}$, e_{2i-1} is said to be *a deletable step in S* if:

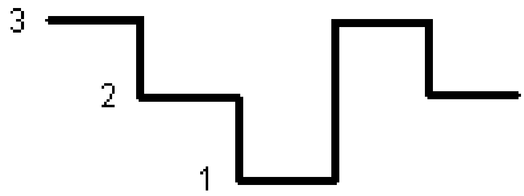


figure 1

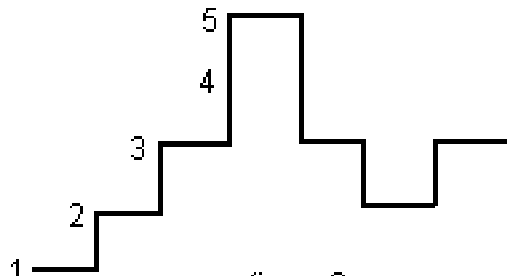


figure 2

(i) there is some $j \neq i$ such that $h_j = h_i$.

(ii) $h_{i-1} \neq h_{i+1}$.

We denote the set of deletable steps in S by D_S.

Definition 5.2.2. Let $S =< h_1, \ldots, h_m >$ be a step diagram with edges $e_1, \ldots, e_{2m-1}$. Suppose e_{2i-1} is a deletable step in S. Then define

$$S - e_{2i-1} =< h_1, \ldots, h_{i-1}, h_{i+1}, \ldots, h_m > .$$

Notice that the criterion of deletability ensures that $h(S) = h(S - e)$ for any $e \in D_S$.

5.3 The definition of $\mathcal{S}$

For each integer $k \geq 2, n \geq 0$, let $\mathcal{S}(k)_n$ be the free $\mathbb{Z}_2$-module generated by all the step diagrams of height $k-1$ with $k+n$ steps. Let $d : \mathcal{S}(k)_n \to \mathcal{S}(k)_{n-1}$

be given by

$$dS = \sum_{e \in D_S} S - e.$$

Let $\mathcal{S}(1)$ be the free graded $\mathbb{Z}_2$-module generated by the unique step diagram of height 0 concentrated in degree 0.

Lemma 5.3.1. *$d \circ d : \mathcal{S}(k)_n \to \mathcal{S}(k)_{n-2}$ is the zero map for all $n \geq 2$.*

Proof. Let $S \in \mathcal{S}(k)_n$ be a step diagram. Then

$$d \circ dS = \sum_{e \in D_S} \sum_{f \in D_{S-e}} (S - e) - f. \tag{5.1}$$

We show that the each summand in the expression on the right hand side of (5.1) can be 'paired off' with a summand of equal value. Since the coefficient ring is $\mathbb{Z}_2$, this shows that they sum to zero.

Suppose the edges of $S =< h_1, \ldots, h_m >$ are $e_1, \ldots, e_{2m-1}$. Let e_{2i-1} be a deletable step in S and e_{2j-1} a deletable step in $S - e_{2i-1}$. Then e_{2j-1} is a deletable step in S so we can form $S - e_{2j-1}$. We want to show that e_{2i-1} is deletable in $S - e_{2j-1}$. If $h_j \neq h_i$ and $|i - j| \neq 1$ then e_{2i-1} is automatically deletable in $S - e_{2j-1}$. If $h_i = h_j$ then since e_{2j-1} is deletable in $S - e_{2i-1}$, there is another edge f in S such that $f \neq e_{2i-1}$ and $f \neq e_{2j-1}$ but $h(f) = h_i$. Therefore f is an edge in $S - e_{2j-1}$ such that $h_i = h(f)$. It follows that e_{2i-1} is deletable in $S - e_{2j-1}$ as long as the two edges to the left and right of e_{2i-1} in $S - e_{2j-1}$ are not of the same height. But since e_{2i-1} is deletable in S, this can only happen when $|i - j| = 1$, which can never be the case if $h_i = h_j$.

Now suppose $|i - j| = 1$ but $h_j \neq h_i$. Without loss of generality, assume $i = j + 1$. If $h_{j-1} = h_{i+1}$ then e_{2j-1} is not a deletable step in $S - e_{2i-1}$, contradicting our original assumptions. Therefore $h_{j-1} \neq h_{i+1}$ so e_{2i-1} is deletable in $S - e_{2j-1}$.

Therefore we can form $(S - e_{2j-1}) - e_{2i-1}$. Furthermore, $(S - e_{2j-1}) - e_{2i-1} = (S - e_{2i-1}) - e_{2j-1}$ so $(S - e_{2j-1}) - e_{2i-1}$ and $(S - e_{2i-1}) - e_{2j-1}$ cancel one another out on the right hand side of (5.1). □

In the light of this lemma, we make the following definition:

Definition 5.3.1. Let $\mathcal{S}(k)$ denote the differential graded $\mathbb{Z}_2$-vector space which is $\mathcal{S}(k)_n$ in degree n and has differential d.

The rest of section 6.3 involves showing that the various $\mathcal{S}(k)$ fit together to form an operad.

Lemma 5.3.2. *Let k be a positive integer. The set of step diagrams of height $k-1$ admits a free action of the symmetric group Σ_k.*

Proof. For the purposes of this proof, let St_k denote the set of step diagrams of height $k-1$. Let $S =< h_1, \ldots, h_m >$ be a step diagram in St_k. Let $\sigma \in \Sigma_k$. Then define $\sigma(S) =< \sigma(h_1), \ldots, \sigma(h_m) >$. Suppose now that $\sigma \in \Sigma_k$ is not the identity. Then there exists $i \in \{1, \ldots, k\}$ such that $\sigma(i) \neq i$. Since S is a step diagram, there is some j such that $h_j = i$. Therefore $\sigma(h_j) \neq h_j$ so $\sigma(S) \neq S$. Hence the action of Σ_k on St_k is free. □

Corollary 5.3.3. *$\mathcal{S}(k)$ admits a free action of Σ_k.*

Note. Notice that σ is an isomorphism of differential graded vector spaces.

Next we construct operadic structure maps $\chi : \mathcal{S}(n) \otimes \mathcal{S}(k_1) \otimes \cdots \otimes \mathcal{S}(k_n) \to \mathcal{S}(k)$. The formalism for this construction rather masks the essence of it, so before a formal definition is given, we will look at some examples.

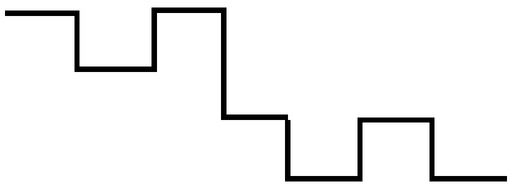
As a first example, we construct the step diagram that models $(a \cup_1 b) \cup_0 (c \cup_2 d)$. Let X be a topological space and f a singular n-simplex with vertices $v_0, \dots, v_n$. Suppose that

$$(a \cup_1 b) \cup_0 (c \cup_2 d)(\phi) = \begin{cases} 1 & \text{if } \phi = f \\ 0 & \text{otherwise} \end{cases}$$

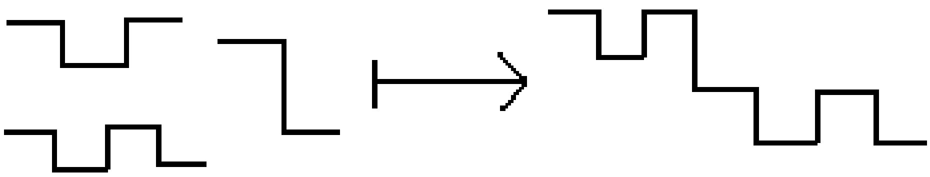
Then a, b, c, d must be dual to singular simplices $\sigma_a, \sigma_b, \sigma_c, \sigma_d$ with vertices that look like:

σ_a	$v_0 \dots v_h$		$v_i \dots v_j$				
σ_b		$v_h \dots v_i$					
σ_c				$v_j \dots v_k$		$v_l \dots v_m$	
σ_d					$v_k \dots v_l$		$v_m \dots v_n$

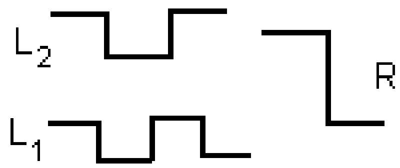
Therefore the step diagram modeling $(a \cup_1 b) \cup_0 (c \cup_2 d)$ should look like:



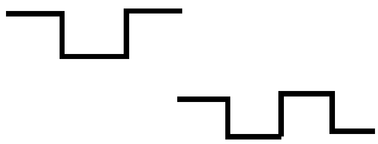
Let $S(i)$ be the step diagram representing the cup-i product for every $i \geq 0$. The above step diagram ought to be the image of $S(0) \otimes (S(1) \otimes S(2))$ in the map χ. Therefore, in this example, χ can be drawn as follows:



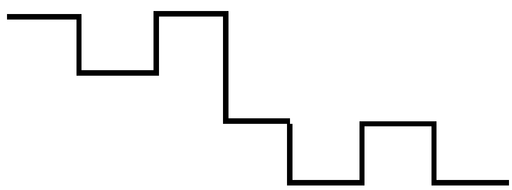
At this point, it is worth examining how to construct χ in this example without reference to specific singular cochains. We start by looking at the left-hand side of the above map:



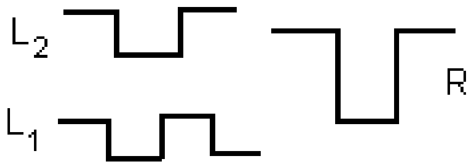
The idea is to combine the two left-hand step diagrams *in the way suggested by the right-hand step diagram*. More precisely, we do the following. Let R denote the right-hand step diagram. Let L_1, L_2 denote the bottom-left and top-left step diagrams respectively. Look at the first step of R. This has height 2, which tells us to start off by drawing L_2. Since there is only one step of height 2 in R, we draw the whole of L_2. Now look at the next step of R. This has height 1 so now we draw L_1. Since there is only one step of height 1 in R, we draw the whole of L_1. This is drawn below and to the right of the first part. So far, our drawing should look like:



The two sections should then be joined by a vertical line to give:

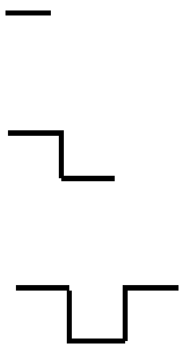


As another example, we look at $(a \cup_1 b) \cup_1 (c \cup_2 d)$. We start off with the following diagram:

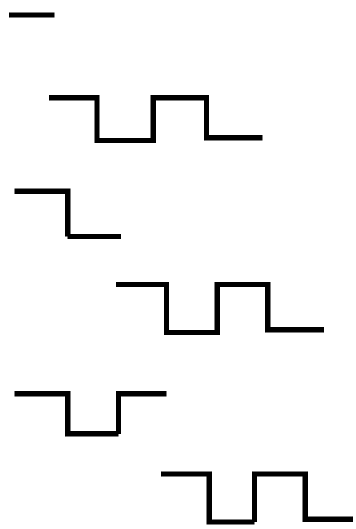


The first step of R has height 2 so we start off by looking at L_2. There are two steps of height 2 in R, so we need to split L_2 into two. This can be done in three possible ways:

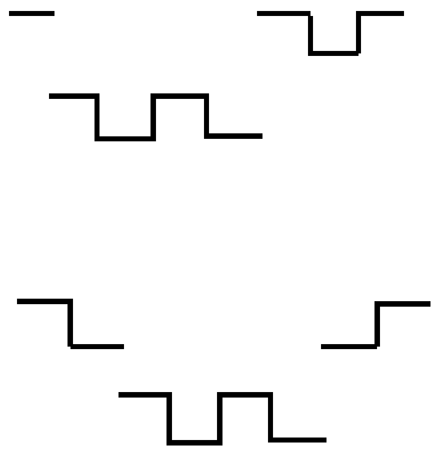
Therefore we start by drawing three diagrams consisting of the three possible left-hand sections of L_2:

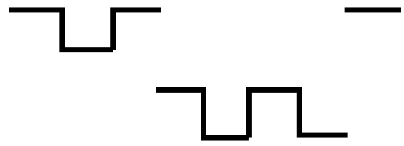


Next, we look at the second step of R. This has height 1. There is only one step of height 1, so no need to split up L_1. Therefore we draw L_1 below and to the right of each of our drawings:

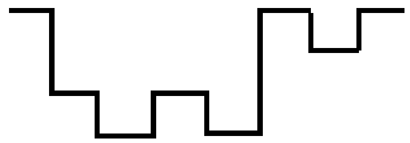


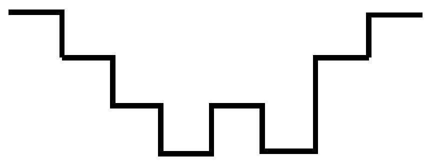
Next, we look at the third step of R. This is the other step of height 2 so we draw the remaining sections of L_2 above and to the right of the three drawings:

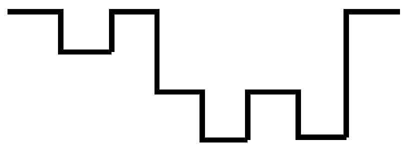




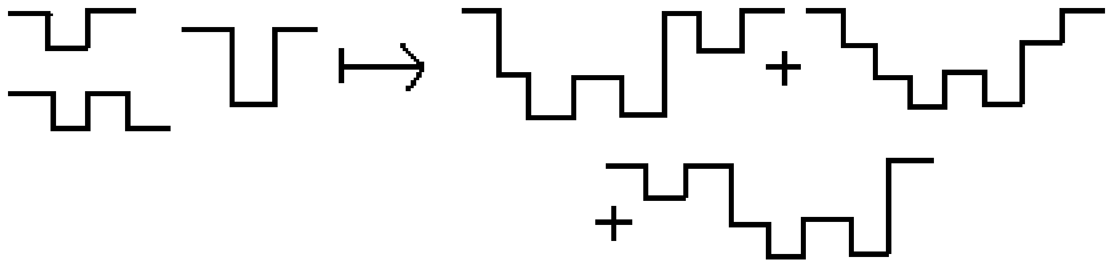
To complete the picture, we just join up the relevant sections:







The sum of these is the image of $S(1)\otimes(S(1)\otimes S(2))$ in the map χ. Therefore, in this example, χ can be drawn as follows:



The reader may wish to examine several other examples in this way until the definition of χ becomes apparent, at least on an intuitive level. Having done this, we begin formulating a formal definition.

Definition 5.3.2. Let $S =< h_1, \ldots, h_n >$ be a step diagram and m a non-negative integer. An *m-partition of S* is a collection of pre-step diagrams $S_1, \ldots, S_{m+1}$ such that $S_1 =< h_1, \ldots, h_{n_1} >, S_2 =< h_{n_1}, \ldots, h_{n_2} >, \ldots, S_{m+1} =< h_{n_m}, \ldots, h_n >$ for some $1 \leq n_1 \leq \cdots \leq n_m \leq n$.

Let $S =< h_1, \ldots, h_n >$ be a step diagram. Let P be an m-partition of S. Defining P is equivalent to defining a set of integers $1 \leq n_1 \leq \cdots \leq n_m \leq n$. Therefore instead of defining P the collection of pre-step diagrams $S_1, \ldots, S_{m+1}$, we can define P to be the collection of integers $n_1, \ldots, n_m$. Such a definition is useful for calculations.

Definition 5.3.3. Let S be a step diagram and m a non-negative integer. Let $1 \leq n_1 \leq \cdots \leq n_m \leq n$ be an m-partition of S. Let $k < m$. A *k-subpartition of P* is a subset X of $\{n_1, \ldots, n_m\}$. Denote this subpartition by K. The $m-k$-subpartition of P given by the subset $\{n_1, \ldots, n_m\} - X$ of $\{n_1, \ldots, n_m\}$ is denoted K^c.

Definition 5.3.4. Let T be a step diagram of height $n-1$. For each i, let $m_i(T)$ be such that $m_i(T)+1$ is the number of horizontal edges of T with $h(e) = i$. Define a *T-partition of* $S_1, \ldots, S_n$ to be an $m_i(T)$-partition of S_i

for each i. If $P_1, \ldots, P_n$ are $m_i(T)$-partitions of $S_1, \ldots, S_n$ respectively then $< P_1, \ldots, P_n >$ is used to denote the corresponding T-partition of $S_1, \ldots, S_n$. Let $P(T; S_1, \ldots, S_n)$ denote the set of T-partitions of $S_1, \ldots, S_n$.

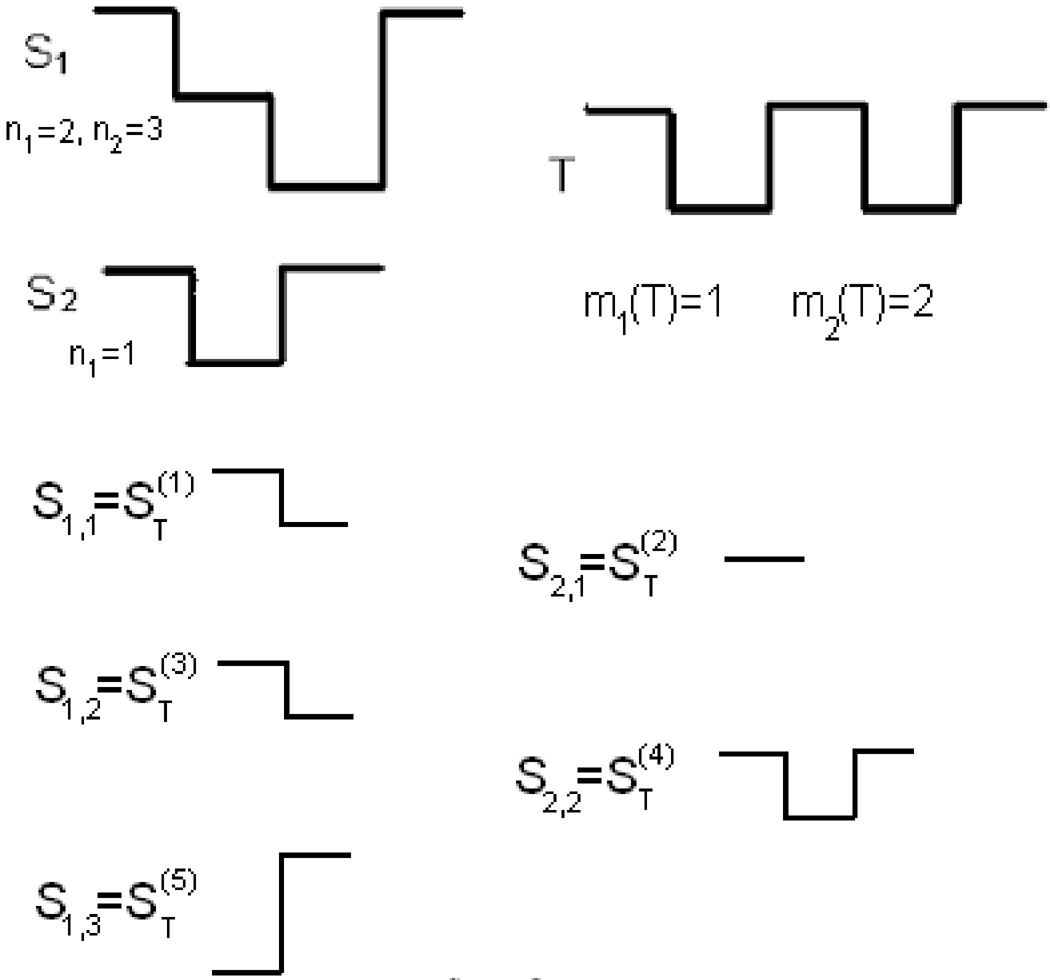


figure 3

Let $T, S_1, \ldots, S_n$ be step diagrams such that $h(T) = n - 1$. Fix a T-partition P of $S_1, \ldots, S_n$ so that for each i, $S_{i,1}, \ldots, S_{i,m_i}$ is the corresponding m_i-partition of S_i.

Let $T =< h_1, \ldots, h_m >$. Note that $m = m_1(T) + \cdots + m_n(T) + n$. Let $S_T^{(1)} = S_{h_1,1}$ and inductively define $S_T^{(t)}$ to be either:

(i) $S_{h_t,1}$ if $h_s \neq h_t$ for every $s < t$.

(ii) $S_{h_t,a+1}$ otherwise, where a is the number of elements of $\{h_1, \ldots, h_{t-1}\}$ that are equal to h_t.

See figure 3 for an example picture of this situation.

For each t, define an integer L_t as follows. If $h_t = h_{t+1}$ then let $L_t = 0$. If $h_t > h_{t+1}$ then let

$$L_t = -\sum_{i=h_{t+1}}^{h_t - 1} (h(S_i) + 1).$$

If $h_{t+1} > h_t$ then let

$$L_t = \sum_{i=h_t}^{h_{t+1}-1} (h(S_i) + 1).$$

Definition 5.3.5. Let $A =< g_1, \ldots, g_a >$, $B =< h_1, \ldots, h_b >$ be pre-step diagrams. Let $L \in \mathbb{Z}$. The pre-step diagram $A +_L B$ is defined to be $< g_1, \ldots, g_a, h_1 + L, \ldots, h_b + L >$ if $L \geq 0$ and $< g_1 - L, \ldots, g_a - L, h_1, \ldots, h_b >$ if $L \leq 0$.

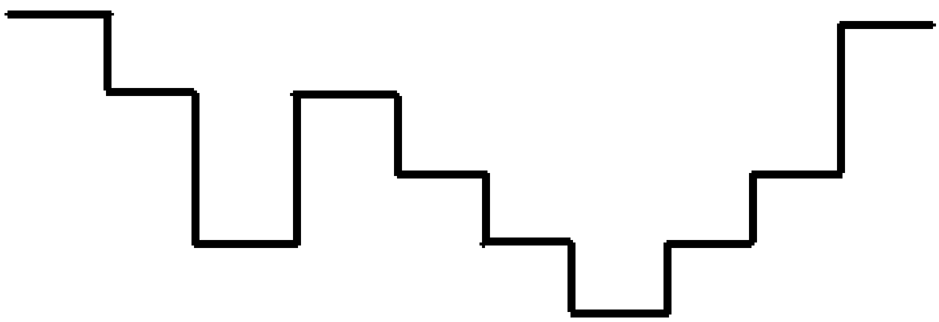


figure 4

Then define

$$S(T;P) := S_T^{(1)} +_{L_1} S_T^{(2)} +_{L_2} \cdots +_{L_{m-1}} S_T^{(m)}.$$

A picture of $S(T;P)$ is given in figure 4 in the case where T and P are as in figure 3. Notice that $S(T;P)$ will always be a step diagram, despite the fact that it is constructed by concatenating pre-step diagrams.

Definition 5.3.6. For each collection of strictly positive integers $n, k_1, \ldots, k_n$, let $k = k_1 + \cdots + k_n$ and define

$$\chi : \mathcal{S}(n) \otimes \mathcal{S}(k_1) \otimes \cdots \otimes \mathcal{S}(k_n) \to \mathcal{S}(k)$$

$$\chi(T \otimes S_1 \otimes \cdots \otimes S_n) = \sum_{P \in P(T;S_1,\ldots S_n)} S(T;P)$$

for step diagrams $T, S_1, \ldots, S_n$. We also use χ to denote the sum of the various χ over all possible $n, k_1, \ldots, k_n$. The ambiguity will be removed by the context.

Lemma 5.3.4. *χ is a map of differential graded vector spaces.*

Proof. Let $T \in \mathcal{S}(n), S_i \in (k_i)$ be step diagrams. We need to show:

$$\chi(dT \otimes S_1 \otimes \cdots \otimes S_n) + \sum_{i=1}^{n} \chi(T \otimes S_1 \otimes \cdots \otimes S_{i-1} \otimes dS_i \otimes S_{i+1} \otimes \cdots \otimes S_n)$$

$$= d \circ \chi(T \otimes S_1 \otimes \cdots \otimes S_n).$$

That is, if

$$\Psi_1 = \sum_{P \in P(T;S_1,\ldots,S_n)} \sum_{e \in D_{S(T;P)}} S(T;P) - e$$

$$\Psi_2 = \sum_{i=1}^{n} \sum_{e \in D_{S_i}} \sum_{P \in P(T;S_1,\ldots,S_{i-1},S_i-e,S_{i+1},\ldots,S_n)} S(T;P)$$

$$\Psi_3 = \sum_{e\in D_T} \sum_{P\in P(T-e;S_1,\dots,S_n)} S(T-e;P)$$

then

$$\Psi_1 + \Psi_2 + \Psi_3 = 0. \tag{5.2}$$

The tactic is to 'pair off' each summand in the expansions of Ψ_1, Ψ_2, Ψ_3 with another summand of equal value. Then the fact that the ground field is $\mathbb{Z}_2$ will give the desired result.

Summands in Ψ_1. First we look at the summands in the expansion of Ψ_1. Fix a T-partition P of $S_1, \dots, S_k$. Let e be a deletable edge in $S(T;P)$. Then $S(T;P)-e$ is a summand in (5.2). For each p, let $e_1^{(p)}, \dots, e_{2s_p-1}^{(p)}$ be the edges of $S^{(p)}$ and f_p a vertical edge of length L_p. Then $S(T;P)$ has the following ordered set of edges:

$$\{e_1^{(1)}, \dots, e_{2s_1-1}^{(1)}, f_1, e_1^{(2)}, \dots, e_{2s_2-1}^{(2)}, f_2, \dots\dots, e_1^{(m)}, \dots, e_{2s_m-1}^{(m)}\}.$$

Let $h_i^{(j)}$ denote the height of $e_{2i-1}^{(j)}$ in $S(T;P)$. Then

$$S(T;P) =< h_1^{(1)}, \dots, h_{s_1}^{(1)}, h_1^{(2)}, \dots, h_{s_2}^{(2)}, \dots\dots, h_1^{(m)}, \dots, h_{s_m}^{(m)} > .$$

There are four cases to consider:

(i) $e \notin \{e_{2s_1-1}^{(1)}, \dots, e_{2s_{m-1}-1}^{(m-1)}\} \cup \{e_1^{(2)}, \dots, e_1^{(m)}\}$.

(ii) $e = e_{2s_p-1}^{(p)}$ and $s_p = 1$.

(iii) $e = e_{2s_p-1}^{(p)}$ and $s_p \neq 1$.

(iv) $e = e_1^{(p)}$ and $s_p \neq 1$.

Case (i). Let $1 \le n_{i,1} \le \cdots \le n_{i,m_i}$ be the m_i-partition of S_i used to construct the T-partition P. In this case, e is not the $n_{i,j}$-th step of S_i for any $1 \le j \le m_i$. Therefore $n_{i,1} \le \cdots \le n_{i,m_i}$ is an m_i-partition of $S - e$.

Denote this partition by $\bar{P}_i$. Let $\bar{P}$ be the T-partition of $S_1, \ldots, S_{i-1}, S_i - e, S_{i+1}, \ldots, S_n$ given by $\bar{P} =< P_1, \ldots, P_{i-1}, \bar{P}_i, P_{i+1}, \ldots, P_n >$. Then $S(T; \bar{P})$ is a summand in Ψ_2 and $S(T; \bar{P}) = S(T; P) - e$. Therefore $S(T; \bar{P})$ and $S(T; P) - e$ cancel one another out on the left hand side of (5.2), so we pair them off.

Case (ii). Here, $S(T;P)-e =< h_1^{(1)}, \ldots, h_{s_1}^{(1)}, \ldots\ldots, h_1^{(p-1)}, \ldots, h_{s_{p-1}}^{(p-1)}, h_1^{(p+1)}, \ldots, h_{s_{p+1}}^{(p+1)}, \ldots\ldots, h_1^{(m)}, \ldots, h_{s_m}^{(m)} >$. Let i, j be such that $S_T^{(p)} = S_{i,j}$ (recalling the notation from before). Let $\bar{P}_i$ be the $(m_i - 1)$-partition of S_i given by the collection of pre-step diagrams $S_{i,1}, \ldots, S_{i,j-1}, S_{i,j} +_0 S_{i,j+1}, S_{i,j+2}, \ldots, S_{i,m_i}$. Let $\bar{P} =< P_1, \ldots, P_{i-1}, \bar{P}_i, P_{i+1}, \ldots, P_n >$. Then $\bar{P} \in P(T - e_{2p-1}; S_1, \ldots, S_n)$ so we can form $S(T - e_{2p-1}; \bar{P})$. Further, $S(T - e_{2p-1}; \bar{P})$ is a summand in Ψ_3 and $S(T - e_{2p-1}; \bar{P}) = S(T; P) - e$ so these two summands cancel one another out on the left hand side of (5.2), so we pair them off.

Case (iii). $S(T; P) - e =< h_1^{(1)}, \ldots, h_{s_1}^{(1)}, \ldots\ldots, h_1^{(p-1)}, \ldots, h_{s_{p-1}}^{(p-1)}, h_1^{(p)}, \ldots, h_{s_p-1}^{(p)}, h_1^{(p+1)}, \ldots, h_{s_{p+1}}^{(p+1)}, \ldots\ldots, h_1^{(m)}, \ldots, h_{s_m}^{(m)} >$. Let i, j be such that $S_T^{(p)} = S_{i,j}$. Using the notation from definition 5.3.2, we see that $S_{i,j} =< h_{n_{j-1}}, \ldots, h_{n_j} >$. Let $\bar{S}_{i,j} =< h_{n_{j-1}}, \ldots, h_{n_j - 1} >$, $\bar{S}_{i,j+1} =< h_{n_j - 1}, \ldots, h_{n_{j+1}} >$. Let $\bar{P}_i$ be the m_i-partition of S_i given by the collection of pre-step diagrams $S_{i,1}, \ldots, S_{i,j-1}, \bar{S}_{i,j}, \bar{S}_{i,j+1}, S_{i,j+2}, \ldots, S_{i,m_i}$. Let $\bar{P} =< P_1, \ldots, \bar{P}_i, \ldots, P_n >$. Then $\bar{P} \in P(T; S_1, \ldots, S_k)$ so we can form $S(T; \bar{P})$. Let $\{\bar{e}_1^{(1)}, \ldots, \bar{e}_{2s_1-1}^{(1)}, f_1, \bar{e}_1^{(2)}, \ldots, \bar{e}_{2s_2-1}^{(2)}, f_2, \ldots\ldots, \bar{e}_1^{(m)}, \ldots, \bar{e}_{2s_m-1}^{(m)}\}$ be the ordered set of edges of $S(T; \bar{P})$, labeled in an analogous way to the edges of $S(T; P)$. Let q be such that $S_T^{(q)} = S_{i,j+1}$. Then $S(T; \bar{P}) - \bar{e}_1^{(q)} = S(T; P) - e$ so these summands cancel one another out on the left hand side of (5.2), so we pair them off.

Case (iv). $S(T; P) - e =< h_1^{(1)}, \ldots, h_{s_1}^{(1)}, \ldots\ldots, h_1^{(p-1)}, \ldots, h_{s_{p-1}}^{(p-1)}, h_3^{(p)}, \ldots,$

$h^{(p)}_{s_p}, h^{(p+1)}_1, \ldots, h^{(p+1)}_{s_{p+1}}, \ldots\ldots, h^{(m)}_1, \ldots, h^{(m)}_{s_m} >$. Let i, j be such that $S^{(p)}_T = S_{i,j}$. Using the notation from definition 5.3.2, we see that $S_{i,j} =< h_{n_{j-1}}, \ldots, h_{n_j} >$. Let $\bar{S}_{i,j} =< h_{n_{j-1}}, \ldots, h_{n_j+1} >$, $\bar{S}_{i,j+1} =< h_{n_j+1}, \ldots, h_{n_{j+1}} >$. Let $\bar{P}_i$ be the m_i-partition of S_i given by the collection of pre-step diagrams $S_{i,1}, \ldots, S_{i,j-1}$, $\bar{S}_{i,j}, \bar{S}_{i,j+1}, S_{i,j+2}, \ldots, S_{i,m_i}$. Let $\bar{P} =< P_1, \ldots, \bar{P}_i, \ldots, P_n >$. Then $\bar{P} \in P(T; S_1, \ldots, S_k)$ so we can form $S(T; \bar{P})$. Let $\{\bar{e}^{(1)}_1, \ldots, \bar{e}^{(1)}_{2s_1-1}, f_1, \bar{e}^{(2)}_1, \ldots, \bar{e}^{(2)}_{2s_2-1}, f_2, \ldots\ldots, \bar{e}^{(m)}_1, \ldots, \bar{e}^{(m)}_{2s_m-1}\}$ be the ordered set of edges of $S(T; \bar{P})$, labeled in an analogous way to the edges of $S(T; P)$. Let q be such that $S^{(q)}_T = S_{i,j-1}$. Then $S(T; \bar{P}) - \bar{e}^{(q)}_{s_{2q-1}} = S(T; P) - e$ so these summands cancel one another out on the left hand side of (5.2). Furthermore, they were paired off in case (iii).

Summands in Ψ_2. Now we deal with the summands in the expansion Ψ_2. Let e_{2a-1} be a deletable edge of S_i. Let P be a T-partition of $S_1, \ldots, S_{i-1}, S_i - e_{2a-1}, S_{i+1}, \ldots, S_n$ and suppose $P =< P_1, \ldots, P_n >$. Let $\bar{P}_i$ be the m_i-partition of S_i given by integers $n_{i,1} \leq \cdots \leq n_{i,m_i}$. Let $\bar{P} =< P_1, \ldots, \bar{P}_i, \ldots, P_n >$. Then $\bar{P}$ is a T-partition of $S_1, \ldots, S_k$ and $S(T; P) = S(T; \bar{P}) - e$ where e is the edge of $S(T; \bar{P})$ corresponding to $e_{2a-1} \in S_i$. Notice that the two summands $S(T; P)$ and $S(T; \bar{P}) - e$ were paired off in case (i) of the section dealing with summands in Ψ_1.

Summands in Ψ_3. Now we deal with the summands in the expansion of Ψ_3. Let e_{2p-1} be a deletable edge of T. Let P be a $T - e_{2p-1}$-partition of $S_1, \ldots, S_n$ and suppose $P =< P_1, \ldots P_n >$. Let $i = h_p$. Then suppose there is some $q > p$ such that $S_{i,j+1} = S^{(q)}_T$ for some j. Let q be the lowest such q and j be such that $S_{i,j+1} = S^{(q)}_T$. If no such q exists then there is some $q < p$ such that $S_{i,j} = S^{(q)}_T$ for some j (else e_{2p-1} would not be deletable). In this case, let q be the greatest possible such q and let j be such that $S_{i,j} = S_T^{(q)}$. Recalling the notation from definition 5.3.2, let $\bar{P}_i$ be the m_i-partition of S_i given by the pre-step diagrams $S_{i,1}, \ldots, S_{i,j}, < h_{n_j} >, S_{i,j+1}, \ldots, S_{i,m_i}$. Then let $\bar{P} =< P_1, \ldots, P_{i-1}, \bar{P}_i, P_{i+1}, \ldots, P_n >$ so $\bar{P}$ is a T-partition of $S_1, \ldots, S_n$. Furthermore, $S(T-e_{2p-1}; P) = S(T;P)-e_1^{(p)}$. Notice that the two summands $S(T - e_{2p-1}; P)$ and $S(T;P) - e_1^{(p)}$ were paired off in case (ii) of the section dealing with summands in Ψ_1.

We conclude that all the summands in the expansion of $\Psi_1 + \Psi_2 + \Psi_3$ cancel one another out so that the sum is zero. □

Lemma 5.3.5. *χ is equivariant with respect to the action of the symmetric group.*

Proof. Let $n, k_1, \ldots, k_n$ be positive integers and $k = k_1 + \cdots + k_n$. Let $\tau \in \Sigma_n$ and $\sigma_i \in \Sigma_{k_i}$ for each $i \in \{1, \ldots, n\}$. There is a natural map $\Sigma_n \times \Sigma_{k_1} \times \cdots \times \Sigma_{k_n} \to \Sigma_k$ (see, for example, [14, section 2.2]). Let π be the image of $(\tau, \sigma_1, \ldots, \sigma_n)$ in this map. Then we need to show that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{S}(n) \otimes \mathcal{S}(k_1) \otimes \cdots \otimes \mathcal{S}(k_n) & \xrightarrow{\chi} & \mathcal{S}(k) \\ \Big\downarrow {\scriptstyle \tau \otimes \sigma_1 \otimes \cdots \otimes \sigma_n} & & \Big\downarrow {\scriptstyle \pi} \\ \mathcal{S}(n) \otimes \mathcal{S}(k_1) \otimes \cdots \otimes \mathcal{S}(k_n) & \xrightarrow{\chi} & \mathcal{S}(k) \end{array}.$$

Let $i \in \{1, \ldots, n\}$. Let e be an edge in S_i. Let $P \in P(T; S_1, \ldots, S_n)$. Then, depending on the partition P, and using the notation on pages 66-67, either:

(i) e is an edge in $S_{i,j}$ for just one j.

(ii) e is an edge in $S_{i,j}$ and $S_{i,j+1}$ for just one j.

We will deal with the case (i) here. The second case is proved similarly. In case (i), there is a unique edge in $S(T;P)$ corresponding to e. We will call this edge e'. Then

$$h(e') = h(e) + \sum_{l=1}^{i-1}(h(S_l)+1).$$

Let e'' be the corresponding edge in $\pi(S(T;P))$. Then

$$h(e'') = \sigma_i(h(e)) + \sum_{l=1}^{\tau(i)-1}(h(S_l)+1).$$

Now let $\bar{P}$ be the $\tau(T)$-partition of $\sigma_1(S_{\tau(1)}),\ldots,\sigma_n(S_{\tau(n)})$ induced by P. Let $\bar{e}$ be the edge in $S(\tau(T);\bar{P})$ corresponding to e. Then

$$h(\bar{e}) = \sigma_i(h(e)) + \sum_{l=1}^{\tau(i)-1}(h(S_l)+1).$$

Since a step diagram is uniquely determined by the heights of its steps, it follows that $\pi(S(T;P)) = S(\tau(T);\bar{P})$. This implies that the diagram above commutes, proving the lemma. □

Lemma 5.3.6. *Let $k, j_1, \ldots, j_k$ be positive integers. Let $j = j_1 + \cdots + j_k$. Let $i_1, \ldots, i_j$ be positive integers. Let $i = i_1 + \cdots + i_j$, $g_s = j_1 + \cdots + j_s$, $h_s = i_{g_{s-1}+1} + \cdots + i_{g_s}$. The following diagram commutes:*

$$\begin{array}{ccc}
\mathcal{S}(k)\otimes\bigotimes_{s=1}^k\mathcal{S}(j_s)\otimes\bigotimes_{r=1}^j\mathcal{S}(i_r) & \xrightarrow{\chi\otimes id} & \mathcal{S}(j)\otimes\bigotimes_{r=1}^j\mathcal{S}(i_r) \\
\downarrow & & \\
\mathcal{S}(k)\otimes\bigotimes_{s=1}^k(\mathcal{S}(j_s)\otimes\bigotimes_{q=1}^{j_s}\mathcal{S}(i_{g_{s-1}+q})) & & \Big\downarrow \chi \\
\Big\downarrow id\otimes\chi^{\otimes k} & & \\
\mathcal{S}(k)\otimes\bigotimes_{s=1}^k\mathcal{S}(h_s) & \xrightarrow{\chi} & \mathcal{S}(i)
\end{array}.$$

Note. This lemma is proved by a direct calculation, but a rather long and cumbersome one. It is not illustrative to justify every step in the calculation so the below proof merely gives an explanation of how to perform the calculation. Details of why the method works are left to the reader. It is helpful to keep an example situation in mind when reading the proof.

Proof. Let $R \in \mathcal{S}(k)$, $T_r \in \mathcal{S}(i_r)$ for each r and $S_s \in \mathcal{S}(j_s)$ for each s be step diagrams. Then

$$\chi \circ (\chi \otimes id)(R \otimes S_1 \otimes \cdots \otimes S_k \otimes T_1 \otimes \cdots \otimes T_j)$$

$$= \sum_{P \in P(R;S_1,\ldots,S_k)} \sum_{Q \in P(S(R;P);T_1,\ldots,T_j)} S(S(R;P);Q) \tag{5.3}$$

$$\chi \circ (id \otimes \chi^{\otimes k})(R \otimes (S_1 \otimes T_1 \otimes \cdots \otimes T_{j_1}) \otimes \cdots \otimes (S_k \otimes T_{j_{k-1}+1} \otimes \cdots \otimes T_{j_k}))$$

$$= \sum_{A_\gamma \in P(S_1;T_{g_{\gamma-1}+1},\ldots,T_{g_\gamma})} \sum_{B \in P(R;S(S_1;A_1),\ldots,S(S_k;A_k))} S(R;B). \tag{5.4}$$

We want to show that we can pair each summand X in the right hand side of (5.3) with a summand Y in the right hand side of (5.4) such that $X = Y$. A summand in (5.3) is obtained by first picking $P \in P(R; S_1, \ldots, S_k)$ and then $Q \in P(S(R;P); T_1, \ldots, T_j)$. Picking such a P is equivalent to picking, for each $s \in \{1, \ldots, k\}$, $\sum_{r=g_{s-1}+1}^{g_s} y_r$ integers $\beta_1^r, \ldots, \beta_{y_r}^r$, such that the height of the β_i^r-th step of S_s is $r - g_{s-1}$. Picking such a Q is equivalent to picking, for each $r \in \{g_{s-1}+1, \ldots, g_s\}$ and each $s \in \{1, \ldots, k\}$, integers $\alpha_1, \ldots, \alpha_{x_r+y_r-1}$, where x_r is the number of steps of height s in T_r. From this information, we demonstrate how to construct $A_s \in P(S_s; T_{g_{s-1}+1}, \ldots, T_{g_s})$ and $B \in P(R; S(S_1; A_1), \ldots, S(S_k; A_k))$ so that $S(R;B) = S(S(R;P);Q)$.

Picking A_s is equivalent to picking $x_r - 1$ integers for each $r \in \{g_{s-1} + 1, \ldots, g_s\}$. In fact, we construct an subpartition K of Q as follows. Suppose

$e_{2a_1-1}, \ldots, e_{2a_b-1}$ are the edges of height $r - g_{s-1}$ and $a_1 < a_2 < \cdots < a_b$. Let $a_{b_\gamma} \in \{a_1, \ldots, a_b\}$ be such that $a_{b_\gamma} = \beta_\gamma$. Suppose:

$$b_1 = b_2 = \cdots = b_{\delta_1} \neq b_{\delta_1+1}$$

$$b_{\delta_1+1} = \cdots = b_{\delta_2} \neq b_{\delta_2+1}$$

$$b_{\delta_2+1} = \cdots = b_{\delta_3} \neq b_{\delta_3+1}$$

$$\cdots\cdots$$

$$b_{\delta_{\epsilon-1}+1} = \cdots = b_{\delta_\epsilon} = b_{y_r}.$$

Then K consists of the following integers:

$$\alpha_1, \ldots, \alpha_{b_1-1}$$

$$\alpha_{b_1+\delta_1}, \ldots, \alpha_{b_{\delta_1+1}+\delta_1-1},$$

$$\alpha_{b_{\delta_1+1}+\delta_2}, \ldots, \alpha_{b_{\delta_2+1}+\delta_2-1},$$

$$\cdots\cdots$$

$$\alpha_{b_{\delta_{\epsilon-2}+1}+\delta_{\epsilon-1}}, \ldots, \alpha_{b_{\delta_{\epsilon-1}+1}+\delta_{\epsilon-1}-1},$$

$$\alpha_{b_{\delta_{\epsilon-1}+1}+\delta_\epsilon}, \ldots, \alpha_{x+y_r-1}.$$

Using this choice of labels, we can form $S(S_s; A_s)$ for each s. Consider the subpartition K^c of T_r. Then the labels given by this subpartition induce labels on the horizontal edges of $S(S_s; A_s)$. These labels can be used to construct an R-partition B of $S(S_1; A_1), \ldots, S(S_k; A_k)$ so that $S(R; B) = S(S(R; P); Q)$. Every such $S(R; B)$ can be constructed in this way from the construction of $S(S(R; P); Q)$. Further, if $P' \in P(R; S_1, \ldots, S_k)$, $Q' \in P(S(R; P); T_1, \ldots, T_j)$ are such that either $P \neq P'$ or $Q \neq Q'$ then the resulting step diagram of $S(R; B)$ is different to the the step diagram $S(R; B)$ constructed using P and Q. $\square$

The following theorem follows immediately from the preceding discussion.

Theorem 5.3.7. *The various $\mathcal{S}(k)$ fit together to give an operad $\mathcal{S}$ with structure map χ.*

Definition 5.3.7. The operad $\mathcal{S}$ is called the *step operad.*

It will be proved in section 6.3 that $\mathcal{S}(k)$ is contractible for every k. Therefore, $\mathcal{S}$ is an E_∞-operad.

5.4 S-regularity for simplices

Let $\sigma_1, \ldots, \sigma_k$ be faces of an N-simplex Δ^N such that σ_i is an n_i-face for each i. Let S be a step diagram. Informally, we wish to say that $\sigma_1, \ldots, \sigma_k$ is an S-regular collection of faces if the vertices of the simplices, when written out as described in pages 59, 67, 68, 'look like' S. The purpose of section 5.4 is to formalise this idea.

Let $n = n_1 + \cdots + n_k - N$. Let $v_0, \ldots, v_N$ be the vertices of Δ^N. We wish to define an ordered set $V_1, \ldots, V_m$ of vertices of Δ^N to keep track of the vertices common to more than one of the faces $\sigma_1, \ldots, \sigma_k$. It will be helpful for future calculations to add $V_0 = v_0$ and $V_{m+1} = v_N$ to this set. So start by letting $V_0 = v_0$. Suppose, inductively, that $V_0, \ldots, V_i$ have been defined and that $V_i = v_s$. If $i = n + k - 1$ or $V_i = v_N$ then stop the induction process. If there exists an integer τ such that $\tau > s$ and $v_\tau \in \sigma_\alpha \cap \sigma_\beta$ for some α, β such that $\alpha \neq \beta$ then let t be the lowest such τ. Otherwise, stop the induction process. Suppose t has been defined. Then there is a maximal subset A of $\{1, \ldots, k\}$ such that $v_t \in \sigma_\alpha$ for each $\alpha \in A$. Let $V_{i+1}, \ldots, V_{i+|A|-1}$ each be equal to v_t. This procedure produces a collection of vertices $V_0, \ldots, V_m$. Let $V_{m+1} = v_N$.

Definition 5.4.1. The collection $V_0, \ldots, V_{m+1}$ is called the collection of *pairwise-common vertices* of $\sigma_1, \ldots, \sigma_k$.

Conventions. Whenever notations described so far in section 5.4 are used, the following conventions will be in place. The vertices of $\sigma_1, \ldots, \sigma_k$ will be given the ordering inherited from the ordering on Δ^N. The sequence $V_i, \ldots, V_j$ means every vertex v_r such that $s \leq r \leq t$ where $V_i = v_s$ and $V_j = v_t$.

Definition 5.4.2. Let $S =< h_1, \ldots, h_{n+k} >$ be a step diagram. The collection $\sigma_1, \ldots, \sigma_k$ is said to be S*-regular* if and only if:

(i) $m = n + k - 1$.

(ii) For every vertex $v \in \Delta^N$, there is an i such that v is a vertex in σ_i.

(iii) For each $j \in \{1, \ldots, n+k\}$, let $\{j_1, \ldots, j_s\}$ be the set of numbers such that $h_{j_1} = \cdots = h_{j_k} = j$. The union of vertices $\bigcup_{i=1}^{s} \{V_{j_i - 1}, \ldots, V_{j_i}\}$ is a disjoint union and is equal to the set of vertices in σ_j.

Note. Condition (i) implies that if S, S' are two step diagrams with different numbers of edges then $\sigma_1, \ldots, \sigma_k$ cannot be both S-regular and S'-regular.

Definition 5.4.3. For each $i \in \{1, \ldots, k\}$, the face of σ_{h_i} spanned by $V_{i-1}, \ldots, V_i$ is called *the face associated to* e_{2i-1}. We also say that σ_{h_i} is *associated to* e_{2i-1}. For each $j \in \{1, \ldots, N\}$, if $v_j \in \{V_{i-1}, \ldots, V_i\}$ for some V_i then we say that $e_{2_i - 1}$ is *an edge associated to* v_j.

To gain an understanding of S-regularity, it is helpful to write the vertices of each face in the shape of the step diagram. For example, suppose $\sigma_1, \sigma_2, \sigma_3$ are faces of Δ^7 and σ_1 is spanned by v_3, v_4, v_5, σ_2 is spanned by v_1, v_2, v_3, v_5, v_6 and σ_3 is spanned by v_0, v_1, v_6, v_7. Let S be the step diagram drawn in figure 5 with edges $e_1, \ldots, e_9$. Then we can write the vertices as follows:

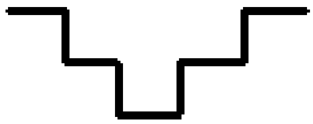


figure 5

σ_3	$v_0\ v_1\ \ \ \ \ \ \ \ \ v_6\ v_7$
σ_2	$v_1\ v_2\ v_3\ \ \ v_5\ v_6$
σ_1	$v_3\ v_4\ v_5$

so that this configuration 'looks like' S. Indeed, it can readily be checked that $\sigma_1, \sigma_2, \sigma_3$ is S-regular. Notice that $V_1 = v_1$, $V_2 = v_3$, $V_3 = v_5$, $V_4 = v_6$.

The following configuration of faces of Δ^6 is also S-regular:

τ_3	$v_0\ v_1\ \ \ \ \ \ \ v_5\ v_6$
τ_2	$v_1\ v_2\ v_3\ \ \ v_5$
τ_1	$v_3\ v_4\ v_5$

However in this case e_7 is associated to only one vertex: v_5. It is possible to be confused into thinking that τ_1, τ_2, τ_3 is $S - e_7$-regular, which the formal definition reveals is not true.

5.5 $S_*(X; \mathbb{Z}_2)$ is a coalgebra over $\mathcal{S}$

Recall that for an operad $\mathcal{O}$ over a unital commutative ring R, we have structure maps

$$\chi : \mathcal{O}(n) \otimes \mathcal{O}(k_1) \otimes \cdots \otimes \mathcal{O}(k_n) \to \mathcal{O}(k)$$

for each collection of positive integers n, $k_1, \ldots k_n$, where $k = k_1 + \cdots + k_n$.

Suppose $\mathcal{O}(1) = R$. For all positive integers k, l define maps

$$\mu_i : \mathcal{O}(l) \otimes \mathcal{O}(k) \to \mathcal{O}(k + l - 1)$$

for each $i \in \{1, \dots, k\}$ by

$$\mu_i(a \otimes b) = \chi(a \otimes 1^{\otimes i-1} \otimes b \otimes 1^{\otimes k-i})$$

where $1 \in \mathcal{O}(1)$ is the unit.

Definition 5.5.1. Let $\mathcal{O}$ be an operad over a unital commutative ring R. Let M be a differential graded R-module. Then M is said to be a *coalgebra over* $\mathcal{O}$ (also called an $\mathcal{O}$*-coalgebra*) if for each k, there are Σ_k-equivariant maps of differential graded R-modules

$$\theta_k : \mathcal{O}(k) \otimes M \to M^{\otimes k}$$

such that the following diagrams commute for each k, l:

$$\begin{array}{ccc}
\mathcal{O}(l)\otimes\mathcal{O}(k)\otimes M & \xrightarrow{\mu_i\otimes id} & \mathcal{O}(k+l-1)\otimes M \\
\downarrow{\scriptstyle id\otimes\theta} & & \\
\mathcal{O}(l)\otimes M^{\otimes k} & & \downarrow{\scriptstyle \theta} \\
\downarrow{\scriptstyle \tau\otimes id^{\otimes k-i+1}} & & \\
M^{\otimes i-1}\otimes\mathcal{O}(l)\otimes M\otimes M^{\otimes k-i} & \xrightarrow[id^{\otimes i-1}\otimes\theta\otimes id^{\otimes k-i}]{} & M^{\otimes k+l-1}
\end{array} \tag{5.5}$$

where $\tau : \mathcal{O}(l) \otimes M^{\otimes i-1} \to M^{\otimes i-1} \otimes \mathcal{O}(l)$ switches $\mathcal{O}(l)$ and $M^{\otimes i-1}$.

Let X be a topological space. Let $S_*(X)$ denote the singular chains on X with coefficients in $\mathbb{Z}_2$. Let $n \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{Z}_{\geq 1}$. Let $n_1, \dots, n_k \in \mathbb{Z}_{\geq 0}$. Let $N = n_1 + \cdots + n_k - n$. Let $\phi_i \in S_{n_i}(X)$ for all $i \in \{1, \dots, k\}$ be such that ϕ_i is a map $\Delta^{n_i} \to X$ from the geometric n_i-simplex to X. Let $S \in \mathcal{S}(k)_n$ be a step diagram.

Definition 5.5.2. $\phi_1, \dots, \phi_k$ is said to be *S-regular* if and only if:

(i) there is a map $\phi : \Delta^N \to X$ such that each ϕ_i factors as $\phi_i \circ f_i$ where $f_i : \Delta^{n_i} \to \Delta^N$ is the inclusion of a face.

(ii) $f_1(\Delta^{n_1}), \dots, f_k(\Delta^{n_k})$ is an S-regular collection of faces of Δ^N.

If $\phi_1, \dots, \phi_k$ is S-regular then denote ϕ by $[S; \phi_1, \dots, \phi_k]$. If $\phi_1, \dots, \phi_k$ is not S-regular then $[S; \phi_1, \dots, \phi_k]$ is defined to be $0 \in S_N(X)$.

For each $N, n \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{Z}_{\geq 1}$, we define a map

$$\theta_{k,N,n} : \mathcal{S}(k)_n \otimes S_N(X) \to \sum_{n_1 + \cdots + n_k = N+n} S_{n_1}(X) \otimes \cdots \otimes S_{n_k}(X)$$

by extending the following map linearly:

$$\theta_{k,N,n}(S \otimes \phi) = \sum_{\phi = [S; \phi_1, \dots, \phi_k]} \phi_1 \otimes \cdots \otimes \phi_k$$

for each step diagram S and singular simplex $\phi : \Delta^N \to X$. If it causes no ambiguity, $\theta_{k,N,n}$ will be denoted θ_k or θ.

Theorem 5.5.1. *$S_*(X)$ is a coalgebra over $\mathcal{S}$ with structure map θ.*

Note. Though this proof is long, most of it deals with a few **Long Lists Of Special Subcases**. To aid the reader, the places where these special subcases are dealt with are pointed out throughout the proof (in bold and title-case). To understand the essence of the proof, these special subcases are probably best omitted on first read.

Proof. Let $N, n \in \mathbb{Z}_{\geq 0}$ and $k \in \mathbb{Z}_{\geq 1}$. Let ∂ denote the differential on $S_*(X)$. During the course of this proof, ∂ will sometimes be decomposed into face maps in the usual way. These will be denoted ∂_j. The symbols used for the various face maps $\Delta^n \to \Delta^{n-1}$ will be $\delta_0, \ldots, \delta_n$, also defined in the usual way (see [22, section 1]). Define three maps:

$$\Psi_1 = \left(\sum_i id^{\otimes i-1} \otimes \partial \otimes id^{\otimes k-i} \right) \circ \theta$$

$$\Psi_2 = \theta \circ (id \otimes \partial)$$

$$\Psi_3 = \theta \circ (d \otimes id).$$

Then we need to show for each step diagram S and singular simplex $\phi : \Delta^N \to X$ that

$$(\Psi_1 + \Psi_2 + \Psi_3)(S \otimes \phi) = 0.$$

Thus for the rest of this proof, fix a step diagram $S =< h_1, \ldots, h_{n+k} >$ that has an ordered set of edges $\{e_1, \ldots, e_{2n+2k-1}\}$. Also fix a singular simplex $\phi : \Delta^N \to X$. Now,

$$(\Psi_1 + \Psi_2 + \Psi_3)(S \otimes \phi) =$$

$$\sum_{\phi=[S;\phi_1,\ldots,\phi_k]} \sum_{j=0}^{n_i} \sum_{i=1}^{k} \phi_1 \otimes \cdots \otimes \phi_{i-1} \otimes \partial\phi_i \otimes \phi_{i+1} \otimes \cdots \otimes \phi_k$$

$$+ \sum_{j=0}^{N} \sum_{\partial_j \phi=[S;\phi_1,\ldots,\phi_k]} \phi_1 \otimes \cdots \otimes \phi_k$$

$$+ \sum_{e \in D_S} \sum_{\phi=[S-e;\phi_1,\ldots,\phi_k]} \phi_1 \otimes \cdots \otimes \phi_k. \tag{5.6}$$

The tactic is to 'pair off' each non-zero summand in the right-hand side of (5.6) with another summand of equal value. Then since the ground field has characteristic 2, the sum will be zero.

Summands arising from Ψ_1. We start with the summands arising from the expansion of $\Psi_1(S \otimes \phi)$. Let $n_1, \ldots, n_k \in \mathbb{Z}_{\geq 0}$ be such that $n_1 + \cdots + n_k =$

$n + N$. For each $i \in \{1, \dots, k\}$, let $\phi_i : \Delta^{n_i} \to X$. Suppose there exists $\phi \in S_N(X)$ such that $\phi = [S; \phi_1, \dots, \phi_k] \neq 0$. Then $\phi_i = \phi \circ f_i$, where $f_i : \Delta^{n_i} \to \Delta^N$ is the inclusion of a face. Let $\sigma_i = f_i(\Delta^{n_i})$. Then $\sigma_1, \dots, \sigma_k$ is S-regular. Let $v_0, \dots, v_N$ be the vertices of Δ^N. Let $V_0, \dots, V_{n+k}$ be the collection of pairwise-common vertices of $\sigma_1, \dots, \sigma_k$. Fix some $i \in \{1, \dots, k\}$ and $j \in \{0, \dots, n_i\}$. Then we are looking either to pair off the summand $\phi_1 \otimes \cdots \otimes \phi_{i-1} \otimes \partial_j \phi_i \otimes \phi_{i+1} \otimes \cdots \otimes \phi_k$ with another summand in the right-hand side of equation (5.6) or to show that $\phi_1 \otimes \cdots \otimes \phi_{i-1} \otimes \partial_j \phi_i \otimes \phi_{i+1} \otimes \cdots \otimes \phi_k = 0$. We look at three cases:

I. σ_i is a single vertex.

II. σ_i contains more than one vertex and the j-th vertex of σ_i is not one of $V_1, \dots, V_{n+k-1}$.

III. σ_i contains more than one vertex and the j-th vertex of σ_i is V_a for some $a \in \{1, \dots, n + k - 1\}$.

Case I. In this case, $n_i = 0$ so $j = 0$ so $\phi_1 \otimes \cdots \otimes \phi_{i-1} \otimes \partial_j \phi_i \otimes \phi_{i+1} \otimes \cdots \otimes \phi_k = 0$.

Case II. First, some clarification: if σ is an oriented simplex spanned by $u_0, \dots, u_p$ then 'the j-th vertex of σ' is defined to be u_j (i.e. we start counting at 0).

For each $l \neq i$, let $\tau_l = \sigma_l$. Let $\tau_i = f_i(\delta_j \Delta^{n_i})$, where $\delta_j \Delta^{n_i}$ is the face of Δ^{n_i} spanned by all the vertices of Δ^{n_i} except the j-th. Let v_α be the j-th vertex of $f_i(\Delta^{n_i})$. Let Δ^{N-1}_α be the $(N-1)$-face of Δ^N spanned by all the vertices except v_α. Then τ_l is a face of Δ^{N-1}_α for each l. Let $u_0, \dots, u_{N-1}$ be the vertices of Δ^{N-1}_α and $U_0, \dots, U_{m+1}$ be the collection of pairwise-common vertices of $\tau_1, \dots, \tau_k$. Then $m = n + k - 1$ and $U_{b-1}, \dots, U_b$ are vertices of $\tau_{h(e_{2b-1})}$ for each b. Hence $\tau_1, \dots, \tau_k$ is S-regular and hence $\phi_1, \dots, \phi_{i-1}, \partial_j \phi_i, \phi_{i+1}, \dots, \phi_k$ is S-regular. Further, $[S; \phi_1, \dots, \phi_{i-1}, \partial_j \phi_i, \phi_{i+1}, \dots, \phi_k] = \partial_\alpha \phi$ so $\theta(S \otimes \partial_\alpha \phi)$ is a summand in the expansion of $\Psi_2(S \otimes \phi)$ that is equal to $\phi_1 \otimes \cdots \otimes \phi_{i-1} \otimes \partial_j \phi_i \otimes \phi_{i+1} \otimes \cdots \otimes \phi_k$. Therefore these two summands are paired off.

Example. Suppose $k = 3$, $N = 8$ and S is the step diagram in figure 5. Suppose the vertices of $\sigma_1, \sigma_2, \sigma_3$ are those suggested by the following diagram:

σ_3	v_0	v_1	v_2					v_7	v_8
σ_2			v_2	v_3			v_6	v_7	
σ_1				v_3	v_4	v_5	v_6		

Then $\sigma_1, \sigma_2, \sigma_3$ is S-regular. Suppose $i = 1$ and $j = 1$. The first vertex of σ_1 is v_4 so this is an example of case II. It can readily be seen that τ_1, τ_2, τ_3 is S-regular.

Case III. This case is one of the **Long Lists Of Special Subcases** referred to just after the statement of theorem 5.5.1.

Suppose the j-th vertex of σ_i is V_a for some a. Let $\alpha \in \{0, \dots, N\}$ be such that $v_\alpha = V_a$. Let $\{i_1, \dots, i_t\} \subseteq \{1, \dots, k\}$ be the largest set such that $v_\alpha \in \sigma_{i_l}$ for every $l \in \{1, \dots, t\}$. For each $l \in \{1, \dots, t\}$, there is a unique step of S associated to a face of σ_{i_l} containing v_α. Let X be the set of all such steps for any l. By S-regularity, there exist p, q such that $X = \{e_{2p-1}, e_{2p+1}, \dots, e_{2q-1}\}$. Relabel the sub-subscripts of $\sigma_{i_1}, \dots, \sigma_{i_t}$ so that σ_{i_l} is associated to $e_{2p+2l-3}$. We have three sub-cases to consider:

III.1. $i \neq i_1, i_t$

III.2. $i = i_1$

III.3. $i = i_t$

Case III.1. Let $\tau_l = f_l(\Delta^{n_l})$ for $l \neq i$ and $\tau_i = f_i(\delta_j \Delta^{n_i})$. Let $U_0, \dots, U_{m+1}$ be the collection of pairwise common vertices of $\tau_1, \dots, \tau_k$. Then $m = n+k-2$ and

(i) $U_{b-1}, \dots, U_b$ are vertices in τ_{h_b} for every $b \in \{1, \dots, r-1\}$

(ii) $U_{b-1}, \dots, U_b$ are vertices in $\tau_{h_{b+1}}$ for every $b \in \{r, \dots, n+k-1\}$

since e_{2r-1} is associated to the face of σ_i consisting of just the vertex v_α. Note that e_{2r-1} is deletable, since otherwise σ_i would be a single vertex, which contradicts the assumptions of case III. Therefore $\tau_1, \dots, \tau_k$ is $S - e_{2r-1}$-regular and

$$[S - e_{2r-1}; \phi_1, \dots, \phi_{i-1}, \partial_j \phi_i, \phi_{i+1}, \dots, \phi_k] = \phi.$$

Therefore $\phi_1 \otimes \cdots \otimes \phi_{i-1} \otimes \partial_j \phi_i \otimes \phi_{i+1} \otimes \cdots \otimes \phi_k$ is a summand in both the expansion of $\Psi_3(S \otimes \phi)$ and the expansion of $\Psi_1(S \otimes \phi)$ so we can pair these two summands off.

Example. Suppose $k = 3$, $N = 7$ and S is the step diagram in figure 5. Let $\{e_1, \dots, e_9\}$ be the ordered set of vertices of S. Suppose the vertices of $\sigma_1, \sigma_2, \sigma_3$ are those suggested by the following diagram:

σ_3	v_0	v_1	v_2				v_6	v_7
σ_2			v_2	v_3			v_6	
σ_1				v_3	v_4	v_5	v_6	

Then $\sigma_1, \sigma_2, \sigma_3$ is S-regular. Suppose $i = 2$ and $j = 2$. The second vertex of σ_2 is v_6 so this is an example of case III.1. It can readily be seen that τ_1, τ_2, τ_3 is $S - e_7$-regular.

Case III.2. Let $\tau_l = f_l(\Delta^{n_l})$ for $l \neq i$ and $\tau_i = f_i(\delta_j \Delta^{n_i})$. Let $U_0, \dots, U_{m+1}$ be the collection of pairwise common vertices of $\tau_1, \dots, \tau_k$. Then $m = n+k-2$ and

(i) $U_{a-1}, \dots, U_a$ are vertices in τ_{h_a} for every $a \in \{1, \dots, p-1\}$

(ii) $U_{p-1}, \dots, v_{\alpha-1}$ are vertices in τ_{h_p} but $U_p \notin \tau_{h_p}$

(iii) $v_\alpha, \dots, U_p$ are vertices in $\tau_{h_{p+1}}$

(iv) $U_{a-1}, \dots, U_a$ are vertices in $\tau_{h_{a+1}}$ for every $a \in \{p+1, \dots, n+k-1\}$

There are three sub-cases to consider:

III.2.a. $v_{\alpha-1}$ is not a vertex in σ_{i_2}.

III.2.b. $v_{\alpha-1}$ is a vertex in σ_{i_2}.

III.2.c. $\alpha = 0$.

Case III.2.a. Suppose v_α is the l-th vertex of σ_{i_2}. Construct a map $\bar{\phi}_{i_2} : \Delta^{n_{i_2}+1} \to X$ which is ϕ_{i_2} on restriction to $\delta_l \Delta^{n_{i_2}+1}$, takes the l-th vertex of $\Delta^{n_{i_2}+1}$ to $\phi(v_{\alpha-1})$ and factors as $\bar{\phi}_{i_2} = \phi \circ f$ where f is the inclusion of a face. Let $\bar{\phi}_i = \partial_j \phi_i$. Let $\bar{\phi}_\gamma = \phi_\gamma$ for $\gamma \neq i, i_2$, $\gamma \in \{1, \dots, k\}$. Then $\bar{\phi}_\gamma = \phi \circ f_\gamma$ where f_γ is the inclusion of a face for each $\gamma \in \{1, \dots, k\}$. Let $\tau_\gamma = im(f_\gamma)$.

Then $\tau_1, \dots, \tau_k$ is S-regular and

$$\bar{\phi}_1 \otimes \cdots \otimes \bar{\phi}_{i_2-1} \otimes \partial_l \bar{\phi}_{i_2} \otimes \bar{\phi}_{i_2+1} \otimes \cdots \otimes \bar{\phi}_k = \phi_1 \otimes \cdots \otimes \phi_{i-1} \otimes \partial_j \phi_i \otimes \phi_{i+1} \otimes \cdots \otimes \phi_k$$

so these two summands can be paired off.

Example. Suppose $k = 3$, $N = 7$ and S is the step diagram in figure 5. Suppose the vertices of $\sigma_1, \sigma_2, \sigma_3$ are those suggested by the following diagram:

σ_3	v_0	v_1	v_2				v_6	v_7
σ_2			v_2	v_3			v_6	
σ_1				v_3	v_4	v_5	v_6	

Then $\sigma_1, \sigma_2, \sigma_3$ is S-regular. Suppose $i = 1$ and $j = 3$. The third vertex of σ_1 is v_6, $i_1 = 1 = i$ and $i_2 = 2$. v_5 is not a vertex in σ_2 so this is an example of case III.2.a. Further, τ_1, τ_2, τ_3 (as constructed above) have vertices suggested by the following diagram:

τ_3	v_0	v_1	v_2				v_6	v_7
τ_2			v_2	v_3		v_5	v_6	
τ_1				v_3	v_4	v_5		

so that τ_1, τ_2, τ_3 is S-regular.

Case III.2.b. Let $e_{2s_1-1}, \ldots, e_{2p-1}$ be the edges associated to $v_{\alpha-1}$. Yet again, we need to consider several subcases:

III.2.b(i). $h_{s_1} \neq i_2$.

III.2.b(ii). $h_{s_1} = i_2$ and $\alpha = 1$.

III.2.b(iii). $h_{s_1} = i_2$, $\alpha > 1$ and $v_{\alpha-2}$ is not a vertex in $\sigma_{h_{s_1}+1}$.

III.2.b(iv). $\alpha > 1$. $h_{s_1} = i_2$. If $e_{2s_t-1}, \ldots, e_{2s_{t-1}-1}$ are the edges associated with v_{a-t} for every $t \in \{2, \ldots, \alpha\}$ then $v_{\alpha-t}$ is a vertex in $\sigma_{h_{s_{t-1}}+1}$ for every $t \in \{2, \ldots, \alpha\}$.

III.2.b(v). $\alpha > 2$. $h_{s_1} = i_2$. There is some $\beta \in \{2, \ldots, \alpha-1\}$ such that $v_{\alpha-t}$ is a vertex in $\sigma_{h_{s_{t-1}}+1}$ for every $t \in \{1, \ldots, \beta\}$ but $v_{\alpha-\beta-1}$ is not a vertex in $\sigma_{h_{s_\beta}+1}$.

Case III.2.b(i). In this case:

(i) $U_{b-1}, \ldots, U_b$ are vertices in τ_{h_b} for every $b \in \{1, \ldots, p-3\}$

(ii) $U_{p-3}, \ldots, v_{\alpha-1}$ are vertices in $\tau_{h_{p-2}}$ if e_{2p-5} exists

(iii) $v_{\alpha-1}$ is a vertex in τ_{h_p}

(iv) $v_{\alpha-1}, v_\alpha$ are vertices in $\tau_{h_{p+1}}$

(v) $v_\alpha, \ldots, U_{p+2}$ are vertices in $\tau_{h_{p+2}}$ if e_{2p+3} exists

(vi) $U_{b-1}, \ldots, U_b$ are vertices in $\tau_{h_{b+1}}$ for every $b \in \{p+2, \ldots, n+k-1\}$

and all the vertices in each of $\tau_1, \ldots, \tau_k$ have been accounted for. Therefore

$\tau_1, \ldots, \tau_k$ is $S - e_{2p-3}$-regular. Hence we can pair $\phi_1 \otimes \cdots \otimes \phi_{i-1} \otimes \partial_j \phi_i \otimes \phi_{i+1} \otimes \cdots \otimes \phi_k$ with the summand in $\Psi_3(S \otimes \phi)$ arising from the fact that $\tau_1, \ldots, \tau_k$ is $S - e_{2p-3}$-regular.

Example. Suppose $k = 3$, $N = 3$ and S is the step diagram in figure 5. Let $\{e_1, \ldots, e_9\}$ be the ordered set of vertices of S. Suppose the vertices of $\sigma_1, \sigma_2, \sigma_3$ are those suggested by the following diagram:

σ_3	$v_0\ v_1\ v_2\ v_3$
σ_2	$v_1\ v_2$
σ_1	$v_1\ v_2$

Then $\sigma_1, \sigma_2, \sigma_3$ is S-regular. Suppose $i = 1$ and $j = 1$. The first vertex of σ_1 is v_2, $i_1 = 1 = i$ and $i_2 = 2$. Since v_1 is a vertex in σ_2, this is an example of case III.2.b. Since $s_1 = 1$, $h_{s_1} = 3 \neq i_2$ so this is an example of case III.2.b(i). The vertices of τ_1, τ_2, τ_3 are those suggested by the following diagram:

τ_3	$v_0\ v_1\ v_2\ v_3$
τ_2	$v_1\ v_2$
τ_1	v_1

so that τ_1, τ_2, τ_3 is $S - e_3$-regular.

Case III.2.b(ii). In this case, the faces of σ_i associated to $e_{2s_1-1}, \ldots, e_{2p-3}$ all consist of just $v_{\alpha-1} = v_0$. Furthermore, $s_1 = 1$. Thus $V_1, \ldots, V_t$ are all v_0 and $e_1, \ldots, e_{2p-1}$ are all of different heights and $h(e_1) = h_{p+1}$. We relabel the U_l as follows. Let $U_0, \ldots, U_p = v_0$ and $U_p + t = V_{p+t+1}$ for each $t \in \{1, \ldots, k-1\}$. Then

$U_{b-1}, \ldots, U_b$ are vertices in $\tau_{h(2a+1)}$ for each $b \in \{1, \ldots, n+k-1\}$

and all the vertices in each of $\tau_1, \ldots, \tau_k$ have been accounted for. Therefore $\tau_1, \ldots, \tau_k$ is $S - e_1$-regular so we can pair $\phi_1 \otimes \cdots \otimes \phi_{i-1} \otimes \partial_j \phi_i \otimes \phi_{i+1} \otimes \cdots \otimes \phi_k$

with the summand in $\Psi_3(S \otimes \phi)$ arising from the fact that $\tau_1, \ldots, \tau_k$ is $S - e_1$-regular.

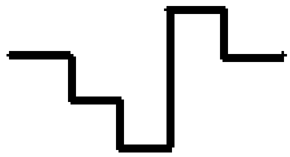


figure 6

Example. Suppose $k = 4$, $N = 3$ and $S =< e_1, \ldots, e_9 >$ is the step diagram in figure 5. Suppose the vertices of $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ are those suggested by the following diagram:

σ_4	v_0 v_1
σ_3	v_0 v_1 v_2 v_3
σ_2	v_0
σ_1	v_0

Then $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ is S-regular. Suppose $i = 4$ and $j = 1$. The first vertex of σ_4 is v_1, $i_1 = 4 = i$ and $i_2 = 3$. Since v_1 is a vertex in σ_1, this is an example of case III.2.b. Since $s_1 = 1$, $h_{s_1} = 3 = i_2$. Also $\alpha = 1$ so this is an example of case III.2.b(ii). The vertices of $\tau_1, \tau_2, \tau_3, \tau_4$ are those suggested by the following diagram:

τ_4	v_0
τ_3	v_0 v_1 v_2 v_3
τ_2	v_0
τ_1	v_0

so that $\tau_1, \tau_2, \tau_3, \tau_4$ is $S - e_1$-regular.

Case III.2.b(iii). In this case:

(i) $V_{b-1}, \ldots, V_b$ are vertices in τ_{h_b} for every $b \in \{2, \ldots, s_1 - 1\}$

(ii) $V_{s_1-1}, \ldots, V_{s_1} = v_{\alpha-1}$ are vertices in $\tau_{h_{s_1}}$

(iii) $v_{\alpha-1}$ is a vertex in τ_{h_b} for every $b \in \{s_1 + 1, \ldots, p\}$

(iv) $V_{b-1}, \ldots, V_b$ are vertices in τ_{h_b} for every $b \in \{p + 1, \ldots, n + k\}$

Let $\kappa = h_{s_1+1}$. Suppose $v_{\alpha-1}$ is the l-th vertex of σ_κ. Let $\bar{\phi}_\kappa : \Delta^{n_\kappa+1} \to X$ be the map such that

(i) $\bar{\phi}_\kappa = \phi \circ f$ where f is the inclusion of a face,

(ii) $\bar{\phi}_\kappa|_{\delta_l \Delta^{n_\kappa+1}} = \phi_\kappa$ and

(iii) $\bar{\phi}_\kappa$ takes the l-th vertex of $\Delta^{n_\kappa+1}$ to $\phi(v_{\alpha-2})$.

Let $\bar{\phi}_i = \partial_j \phi_i$. Let $\bar{\phi}_\gamma = \phi_\gamma$ for $\gamma \neq i, \kappa$. Then we show that $\bar{\phi}_1, \ldots, \bar{\phi}_k$ is S-regular as follows. Each map $\bar{\phi}_\gamma$ can be factored as $\phi \circ \bar{f}_\gamma$ where $\bar{f}_\gamma$ is the inclusion of a face. Let χ_γ be the image of $\bar{f}_\gamma$. Then:

(i) $V_{b-1}, \ldots, V_b$ are vertices in χ_{h_b} for every $b \in \{1, \ldots, s_1 - 1\}$

(ii) $V_{s_1-1}, \ldots, v_{\alpha-2}$ are vertices in $\chi_{h_{s_1}}$

(iii) $v_{\alpha-2}, v_{\alpha-1}$ are vertices in $\chi_{h_{s_1+1}}$

(iv) $v_{\alpha-1}$ is a vertex in χ_{h_b} for each $b \in \{s_1 + 2, \ldots, p\}$

(v) $v_{\alpha-1}, \ldots, V_{p+1}$ are vertices in $\chi_{h_{p+1}}$

(vi) $V_{b-1}, \ldots, V_b$ are vertices in χ_{h_b} for every $b \in \{p + 2, \ldots, n + k\}$

and all the vertices in each of $\chi_1, \ldots, \chi_k$ have been accounted for. Therefore $\bar{\phi}_1, \ldots, \bar{\phi}_k$ is S-regular and

$$\bar{\phi}_1 \otimes \cdots \otimes \bar{\phi}_{\kappa-1} \otimes \partial_l \bar{\phi}_\kappa \otimes \bar{\phi}_{\kappa+1} \otimes \cdots \otimes \bar{\phi}_k = \phi_1 \otimes \cdots \otimes \phi_{i-1} \otimes \partial_j \phi_i \otimes \phi_{i+1} \otimes \cdots \otimes \phi_k$$

so these two summands of $\Psi_1(S \otimes \phi)$ can be paired off.

Example. Suppose $k = 4$, $N = 3$ and $S =< h_1, \ldots, h_5 >$ is the step diagram in figure 6. Suppose the vertices of $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ are those suggested by the following diagram:

σ_4 $v_1\ v_2$

σ_3 $v_0\ v_1\ v_2\ v_3$

σ_2 v_1

σ_1 v_1

Then $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ is S-regular. Suppose $i = 4$ and $j = 1$. The first vertex of σ_4 is v_2, $i_1 = 4 = i$ and $i_2 = 3$. Since v_1 is a vertex in σ_3, this is an example of case III.2.b. Since $s_1 = 1$, $h_{s_1} = 3 = i_2$. Also $\alpha > 1$ and $v_{\alpha-2} = v_0$ is not a vertex in $\sigma_{h_{s_1+1}} = \sigma_{h_2} = \sigma_2$ so this is an example of case III.2.b(iii). The vertices of $\chi_1, \chi_2, \chi_3, \chi_4$ are those suggested by the following diagram:

χ_4 v_1

χ_3 $v_0\ v_1\ v_2\ v_3$

χ_2 $v_0\ v_1$

χ_1 v_1

so that $\chi_1, \chi_2, \chi_3, \chi_4$ is S-regular.

Case III.2.b(iv). In this case:

(i) v_0 is a vertex in τ_{h_b} for every $b \in \{2, \ldots, s_{\alpha-1}\}$

(ii) $v_{\alpha-t-1}$, $v_{\alpha-t}$ are vertices in $\tau_{h_{s_t}}$ for every $t \in \{1, \ldots, \alpha - 1\}$

(iii) $v_{\alpha-t}$ is a vertex in τ_{h_b} for every $b \in \{s_t + 2, \ldots, s_{t-1}\}$ for every $t \in \{2, \ldots, \alpha - 1\}$

(iv) $v_{\alpha-1}$ is a vertex in τ_{h_b} for every $b \in \{s_1 + 2, \ldots, p\}$

(v) $v_2, \ldots, V_{p+1}$ are vertices in $\tau_{h_{p+1}}$

(vi) $V_{b-1}, \ldots, V_b$ are vertices in τ_{h_b}

and all the vertices in each of $\tau_1, \ldots, \tau_k$ have been accounted for. Therefore $\tau_1, \ldots, \tau_k$ is $S - e_1$-regular. Hence we can pair $\phi_1 \otimes \cdots \otimes \phi_{i-1} \otimes \partial_j \phi_i \otimes \phi_{i+1} \otimes \cdots \otimes \phi_k$ with the summand in $\Psi_3(S \otimes \phi)$ arising from the fact that $\tau_1, \ldots, \tau_k$ is $S - e_1$-regular.

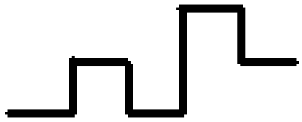


figure 7

Example. Suppose $k = 3$, $N = 3$ and $S =< h_1, \ldots, h_5 >$ is the step diagram in figure 7. Suppose the vertices of $\sigma_1, \sigma_2, \sigma_3$ are those suggested by the following diagram:

σ_3	$\quad v_1\ v_2$
σ_2	$v_0\ v_1\ v_2\ v_3$
σ_1	$v_0\ v_1$

Then $\sigma_1, \sigma_2, \sigma_3$ is S-regular. Suppose $i = 3$ and $j = 1$. The first vertex of σ_3 is v_2, $i_1 = 3 = i$ and $i_2 = 2$. Since v_1 is a vertex in σ_2, this is an example of case III.2.b. Since $s_1 = 2$, $h_1 = 2 = i_2$. Also $\alpha = 2$ and $v_{\alpha-2} = v_0$ is a vertex in $\sigma_{h_{s_1+1}} = \sigma_{h_2} = \sigma_1$ so this is an example of case III.2.b(iv). The vertices of τ_1, τ_2, τ_3 are those suggested by the following diagram:

τ_3	$\quad v_1$
τ_2	$v_0\ v_1\ v_2\ v_3$
τ_1	$v_0\ v_1$

so that τ_1, τ_2, τ_3 is $S - e_1$-regular.

Case III.2.b(v). In this case:

(i) $V_{b-1}, \ldots, V_b$ are vertices in τ_{h_b} for every $b \in \{1, \ldots, s_\beta\}$

(ii) $v_{\alpha-t}$ is a vertex in τ_{h_b} for every $b \in \{s_t + 1, \ldots, s_{t-1} - 1\}$ for every $t \in \{2, \ldots, \beta\}$

(iii) $v_{\alpha-t-1}, v_{\alpha-t}$ are vertices in $\tau_{h_{s_t}}$ for every $t \in \{1, \ldots, \beta - 1\}$

(iv) $v_{\alpha-1}$ is a vertex in τ_{h_b} for every $b \in \{s_1+1,\ldots,p\}$

(v) $v_\alpha,\ldots,V_{p+1}$ are vertices in $\tau_{h_{p+1}}$

(vi) $V_{b-1},\ldots,V_b$ are vertices in τ_{h_b} for every $b \in \{p+2,\ldots,n+k\}$

We split into two further subcases:

III.2.b(v'). There is some $\gamma \in \{1,\ldots,\beta-1\}$ such that $h_{s_\gamma+1} = h_{s_{\gamma+1}+1}$.

III.2.b(v"). No such γ exists.

Case III.2.b(v'). In this case:

(i) $V_{b-1},\ldots,V_b$ are vertices in τ_{h_b} for every $b \in \{1,\ldots,s_\gamma\}$

(ii) $V_{s_\gamma},\ldots,v_{\alpha-\gamma-1}$ are vertices in $\tau_{h_{s_{\gamma+1}}}$

(iii) $v_{\alpha-t-1}$ is a vertex in τ_{h_b} for every $b \in \{s_t+2,\ldots,s_{t-1}\}$ and every $t \in \{2,\ldots,\gamma\}$

(iv) $v_{\alpha-t-1}, v_{\alpha-t}$ are vertices in $\tau_{h_{s_t+1}}$ for every $t \in \{1,\ldots,\gamma\}$

(v) $V_{b-1},\ldots,V_b$ are vertices in τ_{h_b} for every $b \in \{s_1+2,\ldots,p\}$

(vi) $v_{\alpha-1},\ldots,V_{p+1}$ are vertices in $\tau_{h_{p+1}}$

(vii) $V_{b-1},\ldots,V_b$ are vertices in τ_{h_b} for every $b \in \{p+2,\ldots,n+k\}$

and this covers all the vertices in $\tau_1,\ldots,\tau_k$. Therefore $\phi_1,\ldots,\phi_k$ is $S - e_{2s_{\gamma+1}+1}$-regular and so can be paired off with a summand of $\Psi_3(S\otimes\phi)$.

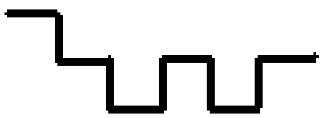


figure 8

Example. Suppose $k=3$, $N=4$ and $S=< h_1,\ldots,h_6 >$ is the step diagram in figure 8. Suppose the vertices of $\sigma_1,\sigma_2,\sigma_3$ are those suggested by the following diagram:

σ_3	v_0	v_1				
σ_2		v_1	v_2	v_3	v_4	
σ_1		v_1	v_2	v_3	v_4	

Then $\sigma_1, \sigma_2, \sigma_3$ is S-regular. Suppose $i = 2$ and $j = 2$. The second vertex of σ_2 is v_3, $i_1 = 2 = i$ and $i_2 = 1$. Since v_2 is a vertex in σ_1, this is an example of case III.2.b. Since $s_1 = 3$, $h_{s_1} = 1 = i_2$. Also $\alpha = 3$ and $v_{\alpha-2} = v_1$ is a vertex in $\sigma_{h_{s_1+1}} = \sigma_{h_4} = \sigma_2$ but $v_{\alpha-3} = v_0$ is not a vertex in $\sigma_{h_{s_2+1}} = \sigma_{h_2} = \sigma_2$ so this is an example of case III.2.b(v) with $\beta = 2$. Since $h_{s_1-1} = h_{s_2-2}$, this is an example of case III.2.b(v') with $\gamma = 1$. The vertices of τ_1, τ_2, τ_3 are those suggested by the following diagram:

τ_3	v_0	v_1			
τ_2		v_1	v_2		v_4
τ_1		v_1	v_2	v_3	v_4

so that τ_1, τ_2, τ_3 is $S - e_3$-regular.

Case III.2.b(v")). Let $\kappa = h_{s_\beta+1}$. Suppose $v_{\alpha-\beta}$ is the l-th vertex of σ_κ. If $\kappa \neq i$ then let $\bar{\phi}_\kappa : \Delta^{n_\kappa+1} \to X$ be the map such that

(i) $\bar{\phi}_\kappa = \phi \circ f$ where f is the inclusion of a face,

(ii) $\bar{\phi}_\kappa|_{\delta_l \Delta^{n_\kappa+1}} = \phi_\kappa$ and

(iii) $\bar{\phi}_\kappa$ takes the l-th vertex of $\Delta^{n_\kappa+1}$ to $\phi(v_{\alpha-\beta-1})$.

and let $\bar{\phi}_i = \partial_j \phi_i$. If $\kappa = i$ then let $\bar{\phi}_\kappa : \Delta^{n_\kappa} \to X$ be the map such that:

(i) $\bar{\phi}_\kappa = \phi \circ f$ where f is the inclusion of a face,

(ii) $\bar{\phi}_\kappa|_{\delta_l \Delta^{n_\kappa}} = \partial_j \phi_k$ and

(iii) $\bar{\phi}_\kappa$ takes the l-th vertex of Δ^{n_κ} to $\phi(v_{\alpha-\beta-1})$.

Let $\bar{\phi}_\gamma = \phi_\gamma$ for $\gamma \neq i, \kappa$. Then every $\bar{\phi}_\gamma$ can be factored as $\bar{\phi}_\gamma = \phi \circ \bar{f}_\gamma$ where

$\bar{f}_\gamma$ is the inclusion of a face. Then

(i) $V_{b-1}, \ldots, V_b$ are vertices in χ_{h_b} for every $b \in \{1, \ldots, s_\beta - 1\}$

(ii) $V_{s_\beta - 1}, \ldots, v_{\alpha-\beta-1}$ are vertices in $\chi_{h(e_{2s_\beta - 1})}$

(iii) $v_{\alpha-t-1}, v_{\alpha-t}$ are vertices in $\chi_{h(e_{2s_t+1})}$ for every $t \in \{1, \ldots, \beta\}$

(iv) $v_{\alpha-t}$ is a vertex in χ_{h_b} for every $b \in \{s_t + 2, \ldots, s_{t-1}\}$ and every $t \in \{2, \ldots, \beta\}$

(v) $v_{\alpha-1}$ is a vertex in χ_{h_b} for every $b \in \{s_1 + 2, \ldots, p\}$

(vi) $v_{\alpha-1}, \ldots, V_{p+1}$ are vertices in $\chi_{h_{p+1}}$

(vii) $V_{b-1}, \ldots, V_b$ are vertices in χ_{h_b} for every $b \in \{p + 2, \ldots, n + k\}$

and all the vertices in each of $\chi_1, \ldots, \chi_k$ have been accounted for. Therefore $\bar{\phi}_1, \ldots, \bar{\phi}_k$ is S-regular and

$$\bar{\phi}_1 \otimes \cdots \otimes \bar{\phi}_{\kappa-1} \otimes \partial_l \bar{\phi}_\kappa \otimes \bar{\phi}_{\kappa+1} \otimes \cdots \otimes \bar{\phi}_k = \phi_1 \otimes \cdots \otimes \phi_{i-1} \otimes \partial_j \phi_i \otimes \phi_{i+1} \otimes \cdots \otimes \phi_k$$

so these two summands of $\Psi_1(S \otimes \phi)$ can be paired off.

Example. Suppose $k = 3$, $N = 4$ and $S =< h_1, \ldots, h_5 >$ is the step diagram in figure 7. Suppose the vertices of $\sigma_1, \sigma_2, \sigma_3$ are those suggested by the following diagram:

σ_3			v_2	v_3		
σ_2		v_1	v_2	v_3	v_4	
σ_1	v_0	v_1	v_2			

Then $\sigma_1, \sigma_2, \sigma_3$ is S-regular. Suppose $i = 3$ and $j = 1$. The first vertex of σ_1 is v_3, $i_1 = 3 = i$ and $i_2 = 2$. Since v_2 is a vertex in σ_2, this is an example of case III.2.b. Since $s_1 = 2$, $h_{s_1} = 2 = i_2$. Also $\alpha = 3$ and $v_{\alpha-2} = v_1$ is a vertex in $\sigma_{h_{s_1+1}} = \sigma_{h_3} = \sigma_1$ but $v_{\alpha-3} = v_0$ is not a vertex in $\sigma_{h_{s_2+1}} = \sigma_{h_2} = \sigma_2$ so this is an example of case III.2.b(v) with $\beta = 2$. Since $h_{s_1-1} \neq h_{s_2-1}$, this is an example of case III.2.b(v"). The vertices of χ_1, χ_2, χ_3 are those suggested by the following diagram:

χ_3			v_2		
χ_2	v_0	v_1	v_2	v_3	v_4
χ_1	v_0	v_1	v_2		

so that χ_1, χ_2, χ_3 is S-regular.

Case III.2.c. In this case $\tau_1, \ldots, \tau_k$ is $S - e_1$-regular so we can pair $\phi_1 \otimes \cdots \otimes \phi_{i-1} \otimes \partial_j \phi_i \otimes \phi_{i+1} \otimes \cdots \otimes \phi_k$ with the summand in $\Psi_3(S \otimes \phi)$ arising from the fact that $\tau_1, \ldots, \tau_k$ is $S - e_1$-regular.

Case III.3. Let $\tau_l = f_l(\Delta^{n_l})$ for $l \neq i$ and $\tau_i = f_i(\delta_j \Delta^{n_i})$. Let $U_0, \ldots, U_{m+1}$ be the collection of pairwise common vertices of $\tau_1, \ldots, \tau_k$. Then $m = n+k-2$ and

(i) $U_{b-1}, \ldots, U_b$ are vertices in τ_{h_b} for every $b \in \{2, \ldots, q-1\}$

(ii) $U_{q-2}, \ldots, v_{\alpha-1}$ are vertices in $\tau_{h(e_{2q-3})}$

(iii) $v_\alpha, \ldots, U_{q-1}$ are vertices in $\tau_{h(e_{2q-1})}$

(iv) $U_{b-1}, \ldots, U_b$ are vertices in τ_{h_b} for every $b \in \{q, \ldots, n+k-1\}$

There are three sub-cases to consider:

III.3.a. $v_{\alpha+1}$ is not a vertex in $\sigma_{i_{t-1}}$

III.3.b. $v_{\alpha+1}$ is a vertex in $\sigma_{i_{t-1}}$

III.3.c. $\alpha = N$

Case III.3.a. Suppose v_α is the l-th vertex of $\sigma_{i_{t-1}}$. Then construct the map $\bar{\phi}_{i_{t-1}} : \Delta^{n_{i_{t-1}}+1} \to X$ with the following properties:

(i) $\bar{\phi}_{i_{t-1}} = \phi \circ f$ where f is the inclusion of a face,

(ii) $\bar{\phi}_{i_{t-1}}|_{\delta_l \Delta^{n_{i_{t-1}}+1}}$ and

(iii) $\bar{\phi}_{i_{t-1}}$ takes the l-th vertex of $\Delta^{n_{i_{t-1}}+1}$ to $\phi(v_{\alpha+1})$.

Let $\bar{\phi}_i = \partial_j \phi_i$. Let $\bar{\phi}_\gamma = \phi_\gamma$ for $\gamma \neq i, i_{t-1}$. Then $\bar{\phi}_1, \ldots, \bar{\phi}_k$ is S-regular and

$$\bar{\phi}_1 \otimes \cdots \otimes \bar{\phi}_{i_{t-1}-1} \otimes \partial_l \bar{\phi}_{i_{t-1}} \otimes \bar{\phi}_{i_{t-1}+1} \otimes \cdots \otimes \bar{\phi}_k = \phi_1 \otimes \cdots \otimes \phi_{i-1} \otimes \partial_j \phi_i \otimes \phi_{i+1} \otimes \cdots \otimes \phi_k$$

so it is possible pair these two summands off. However, notice that these two summands were already paired off in case III.2.a.

Example. Examples of the various subcases of III.3 are very similar to the examples of the subcases of III.2. For instance, an example related to case III.3.a would be as follows. Let S be the vertical-mirror image of figure 5 (which, in this case, is the same as figure 5). Let $\sigma_1, \sigma_2, \sigma_3$ be faces of Δ^7 with vertices as follows:

σ_3	v_7	v_6	v_5				v_1	v_0
σ_2			v_5	v_4			v_1	
σ_1				v_4	v_3	v_2	v_1	

Then $\sigma_1, \sigma_2, \sigma_3$ is S-regular. Suppose $i = 1$ and $j = 0$. The zero-th vertex of σ_1 is v_1, $i_1 = 1 = i$ and $i_2 = 2$. v_2 is not a vertex in σ_2 so this is an example of case III.2.a. Further, τ_1, τ_2, τ_3 (as constructed above) have vertices suggested by the following diagram:

τ_3	v_7	v_6	v_5				v_1	v_0
τ_2			v_5	v_4		v_2	v_1	
τ_1				v_4	v_3	v_2		

so that τ_1, τ_2, τ_3 is S-regular. Compare this with the example in case III.2.a. All the sub-cases of case III.3 are, in a similar way, 'mirror images' of subcases of case III.2. Therefore no futher explicit examples are given for the subcases of III.3. Instead, the reader is referred to the analogous subcases of III.2.

Case III.3.b. Let $e_{2q-1}, \ldots, e_{2s_1-1}$ be the edges associated to $v_{\alpha+1}$. There are five subcases to consider:

III.3.b(i). $h_{s_1} \neq i_{t-1}$

III.3.b(ii). $h_{s_1} = i_{t-1}$ and $\alpha = N-1$

III.3.b(iii). $h_{s_1} = i_{t-1}$, $\alpha < N-1$ and $v_{\alpha+2} \notin \sigma_{h_{s_1-1}}$

III.3.b(iv). $\alpha < N-1$. $h_{s_1} = i_{t-1}$. If $e_{2s_{t-1}-1}, \ldots, e_{2s_t-1}$ are the edges associated with $v_{\alpha+t}$ for every $t \in \{2, \ldots, N-\alpha\}$ then $v_{\alpha+t} \in \sigma_{h_{s_{t-1}-1}}$ for every $t \in \{2, \ldots, N-\alpha\}$.

III.3.b(v). $\alpha < N-2$. $h_{s_1} = i_{t-1}$. There is some $\beta \in \{2, \ldots, N-\alpha-1\}$ such that $v_{\alpha+t} \in \sigma_{h_{s_{t-1}-1}}$ for every $t \in \{2, \ldots, \beta\}$ but $v_{\alpha+\beta+1} \notin \sigma_{h_{s_\beta-1}}$.

Case III.3.b(i). In this case:

(i) $U_{b-1}, \ldots, U_b$ are vertices in τ_{h_b} for every $b \in \{1, \ldots, q-2\}$

(ii) $v_\alpha, v_{\alpha+1}$ are vertices in $\tau_{h_{q-1}}$

(iii) $v_{\alpha+1}$ is a vertex in $\tau_{h(e_{2q-1})}$

(iv) $v_{\alpha+1}, \ldots, U_{q+1}$ are vertices in $\tau_{h_{q+2}}$

(v) $U_{b-1}, \ldots, U_b$ are vertices in τ_{h_b} for every $b \in \{q+2, \ldots, n+k-2\}$

and all the vertices in each of $\tau_1, \ldots, \tau_k$ have been accounted for. Therefore $\phi_1, \ldots, \phi_{i-1}, \partial_j\phi_i, \phi_{i+1}, \ldots, \phi_k$ is $S - e_{2q+1}$-regular and so can be paired off with a summand of $\Psi_3(S \otimes \phi)$.

Case III.3.b(ii). In this case:

(i) $U_{b-1}, \ldots, U_b$ are vertices in τ_{h_b} for every $b \in \{1, \ldots, q-2\}$

(ii) $U_{q-2}, \ldots, v_{\alpha+1}$ are vertices in $\tau_{h_{q-1}}$

(iii) $v_{\alpha+1}$ is a vertex in τ_{h_b} for every $b \in \{q, \ldots, n+k-1\}$

and all the vertices in each of $\tau_1, \ldots, \tau_k$ have been accounted for. Therefore $\phi_1, \ldots, \phi_{i-1}, \partial_j\phi_i, \phi_{i+1}, \ldots, \phi_k$ is $S - e_{2n+2k-1}$-regular and so can be paired off with a summand of $\Psi_3(S \otimes \phi)$.

Case III.3.b(iii). In this case:

(i) $V_{b-1}, \ldots, V_b$ are vertices in τ_{h_b} for every $b \in \{1, \ldots, q-2\}$

(ii) $V_{q-2}, \ldots, v_\alpha$ are vertices in $\tau_{h_{q-1}}$

(iii) $v_{\alpha+1}$ is a vertex in τ_{h_q}

(iv) $v_{\alpha+1}, \ldots, V_{q+1}$ are vertices in $\tau_{h_{q+1}}$

(v) $V_{b-1}, \ldots, V_b$ are vertices in τ_{h_b} for every $b \in \{q+2, \ldots, n+k-1\}$.

Let $\kappa = h_{s_1-1}$. Suppose $v_{\alpha+1}$ is the l-th vertex of σ_κ. Let $\bar{\phi}_\kappa : \Delta^{n_\kappa+1} \to X$ be the map such that:

(i) $\bar{\phi}_\kappa = \phi \circ f$ where f is the inclusion of a face,

(ii) $\bar{\phi}_\kappa|_{\delta_{l+1}\Delta^{n_\kappa+1}} = \phi_\kappa$ and

(iii) $\bar{\phi}_\kappa$ takes the $l+1$-th vertex of $\Delta^{n_\kappa+1}$ to $\phi(v_{\alpha+2})$.

Let $\bar{\phi}_i = \partial_j \phi_i$. Let $\bar{\phi}_\gamma = \phi_\gamma$ for $\gamma \neq i, \kappa$. Then $\bar{\phi}_\gamma = \phi \circ \bar{f}_\gamma$ where $\bar{f}_\gamma$ is the inclusion of a face. Let χ_γ be the image of $\bar{f}_\gamma$. Then:

(i) $V_{b-1}, \ldots, V_b$ are vertices in χ_{h_b} for every $b \in \{1, \ldots, q-3\}$

(ii) $V_{q-2}, \ldots, v_{\alpha+1}$ are vertices in $\chi_{h_{q-1}}$

(iii) $v_{\alpha+1}$ is a vertex in χ_{h_b} for every $b \in \{q, \ldots, s_1-2\}$

(iv) $v_{\alpha+1}, v_{\alpha+2}$ are vertices in $\chi_{h_{s_1-1}}$

(v) $v_{\alpha+2}, \ldots, V_{s_1}$ are vertices in $\chi_{h_{s_1}}$

(vi) $V_{b-1}, \ldots, V_b$ are vertices in χ_{h_b} for every $b \in \{s_1+1, \ldots, n+k\}$

and all the vertices in each of $\chi_1, \ldots, \chi_k$ have been accounted for. Therefore $\bar{\phi}_1, \ldots, \bar{\phi}_k$ is S-regular and

$$\bar{\phi}_1 \otimes \cdots \otimes \bar{\phi}_{\kappa-1} \otimes \partial_l \bar{\phi}_\kappa \otimes \bar{\phi}_{\kappa+1} \otimes \cdots \otimes \bar{\phi}_k = \phi_1 \otimes \cdots \otimes \phi_{i-1} \otimes \partial_j \phi_i \otimes \phi_{i+1} \otimes \cdots \otimes \phi_k.$$

Further, these two summands of $\Psi_1(S \otimes \phi)$ were paired off in case III.2.b(iii).

Case III.3.b(iv). In this case:

(i) $V_{b-1}, \ldots, V_b$ are vertices in τ_{h_b} for every $b \in \{1, \ldots, q-2\}$

(ii) $V_{q-2}, \ldots, v_{\alpha+1}$ are vertices in $\tau_{h_{q-1}}$

(iii) $v_{\alpha+1}$ is a vertex in τ_{h_b} for every $b \in \{q, \ldots, s_1 - 2\}$

(iv) $v_{\alpha+t}, v_{\alpha+t+1}$ are vertices in $\tau_{h_{s_t-1}}$ for every $t \in \{1, \ldots, N-\alpha-1\}$

(v) $v_{\alpha+t}$ is a vertex in τ_{h_b} for every $b \in \{s_{t-1}, \ldots, s_t - 2\}$, for every $t \in \{2, \ldots, N-\alpha\}$

(vi) v_N is a vertex in $\tau_{h_{s_{N-\alpha}}-1} = \tau_{h_{n+k-1}}$

and all the vertices in each of $\tau_1, \ldots, \tau_k$ have been accounted for. Therefore $\phi_1, \ldots, \phi_{i-1}, \partial_j\phi_i, \phi_{i+1}, \ldots, \phi_k$ is $S - e_{2n+2k-1}$-regular and so can be paired off with a summand of $\Psi_3(S \otimes \phi)$.

Case III.3.b(v). In this case:

(i) $V_{b-1}, \ldots, V_b$ are vertices in τ_{h_b} for every $b \in \{1, \ldots, q-2\}$

(ii) $V_{q-1}, \ldots, v_\alpha$ are vertices in $\tau_{h(e_{2q-3})}$

(iii) $v_{\alpha+1}$ is a vertex in τ_{h_b} for every $b \in \{q, \ldots, s_1 - 3\}$

(iv) $v_{\alpha+t}, v_{\alpha+t+1}$ are vertices in $\tau_{h_{s_t}}$ for every $t \in \{1, \ldots, \beta - 1\}$

(v) $v_{\alpha+t}$ is a vertex in τ_{h_b} for every $b \in \{s_t + 1, \ldots, s_{t+1} - 1\}$ for every $t \in \{2, \ldots, \beta\}$

(vi) $V_{b-1}, \ldots, V_b$ are vertices in τ_{h_b} for every $b \in \{s_\beta, \ldots, n+k\}$.

We split into two further subcases:

III.3.b(v'). There is some $\gamma \in \{1, \ldots, \beta - 1\}$ such that $h_{s_\gamma - 1} = h_{s_{\gamma+1}-1}$.

III.3.b(v"). No such γ exists.

Case III.3.b(v'). In this case:

(i) $V_{b-1}, \ldots, V_b$ are vertices τ_{h_b} for every $b \in \{1, \ldots, q-2\}$

(ii) $V_{q-2}, \ldots, v_{\alpha+1}$ are vertices $\tau_{h_{q-1}}$

(iii) $V_{b-1}, \ldots, V_b$ are vertices in τ_{h_b} for every $b \in \{q, \ldots, s_1 - 2\}$

(iv) $v_{\alpha+t}, v_{\alpha+t+1}$ are vertices in $\tau_{h_{s_t-1}}$ for every $t \in \{1, \ldots, \gamma\}$

(v) $v_{\alpha+t+1}$ is a vertex in τ_{h_b} for every $h \in \{s_t, \ldots, s_{t+1} - 2\}$ for every

$t \in \{1, \ldots, \gamma\}$

(vi) $v_{\alpha+\gamma+1}, \ldots, V_{s_{\gamma+1}}$ are vertices in $\tau_{h_{s_{\gamma+1}}}$

(vii) $V_{b-1}, \ldots, V_b$ are vertices in τ_{h_b} for every $b \in \{s_{\gamma+1}+1, \ldots, n+k\}$

and all the vertices in each of $\tau_1, \ldots, \tau_k$ have been accounted for. Therefore $\phi_1, \ldots, \phi_k$ is $S - e_{2s_{\gamma+1}-3}$-regular and so can be paired off with a summand of $\Psi_3(S \otimes \phi)$.

Case III.3.b(v"). Let $\kappa = h_{s_\beta - 1}$. Suppose $v_{\alpha+\beta}$ is the l-th vertex of σ_κ. If $\kappa \neq i$ then let $\bar{\phi}_\kappa : \Delta^{n_\kappa+1} \to X$ be the map such that:

(i) $\bar{\phi}_\kappa = \phi \circ f$ where f is the inclusion of a face,

(ii) $\bar{\phi}_\kappa|_{\delta_{l+1}\Delta^{n_\kappa+1}} = \phi_\kappa$ and

(iii) $\bar{\phi}_\kappa$ takes the $l+1$-th vertex of $\Delta^{n_\kappa+1}$ to $\phi(v_{\alpha+\beta+1})$

and let $\bar{\phi}_i = \partial_j \phi_i$. If $\kappa = i$ then let $\bar{\phi}_\kappa : \Delta^{n_\kappa} \to X$ be the map such that:

(i) $\bar{\phi}_\kappa = \phi \circ f$ where f is the inclusion of a face,

(ii) $\bar{\phi}_\kappa|_{\delta_l \Delta^{n_\kappa}} = \partial_j \phi_\kappa$ and

(iii) $\bar{\phi}_\kappa$ takes the l-th vertex of Δ^{n_κ} to $\phi(v_{\alpha+\beta+1})$.

Let $\bar{\phi}_\gamma = \phi_\gamma$ for $\gamma \neq i, \kappa$. Then $\bar{\phi}_\gamma = \phi \circ \bar{f}_\gamma$ where $\bar{f}_\gamma$ is the inclusion of a face. Let χ_γ be the image of $\bar{f}_\gamma$. Then:

(i) $V_{b-1}, \ldots, V_b$ are vertices in χ_{h_b} for every $b \in \{1, \ldots, q-2\}$

(ii) $V_{q-2}, \ldots, v_{\alpha+1}$ are vertices in $\chi_{h_{q-1}}$

(iii) $v_{\alpha+1}$ is a vertex in χ_{h_b} for every $b \in \{q, \ldots, s_1 - 2\}$

(iv) $v_{\alpha+t}$ is a vertex in χ_{h_b} for every $b \in \{s_t, \ldots, s_{t+1} - 2\}$ for every $t \in \{1, \ldots, \beta - 1\}$

(v) $v_{\alpha+t}, v_{\alpha+t+1}$ are vertices in $\chi_{h_{s_t - 1}}$ for every $t \in \{1, \ldots, \beta\}$

(vi) $v_{\alpha+\beta+1}, \ldots, V_{s_\beta}$ are vertices in $\chi_{h_{s_\beta}}$

(vii) $V_{b-1}, \ldots, V_b$ are vertices in χ_{h_b} for every $b \in \{s_\beta + 1, \ldots, n+k\}$

and all the vertices in each of $\chi_1, \ldots, \chi_k$ have been accounted for. Therefore

$\bar{\phi}_1, \ldots, \bar{\phi}_k$ is S-regular and

$\bar{\phi}_1 \otimes \cdots \otimes \bar{\phi}_{\kappa-1} \otimes \partial_l \bar{\phi}_\kappa \otimes \bar{\phi}_{\kappa+1} \otimes \cdots \otimes \bar{\phi}_k = \phi_1 \otimes \cdots \otimes \phi_{i-1} \otimes \partial_j \phi_i \otimes \phi_{i+1} \otimes \cdots \otimes \phi_k.$

Further, these two summands of $\Psi_1(S \otimes \phi)$ were paired off in case III.2.b(v).

Case III.3.c. In this case $\tau_1, \ldots, \tau_k$ is $S - e_{2n+2k-1}$-regular so we can pair $\phi_1 \otimes \cdots \otimes \phi_{i-1} \otimes \partial_j \phi_i \otimes \phi_{i+1} \otimes \cdots \otimes \phi_k$ with the summand in $\Psi_3(S \otimes \phi)$ arising from the fact that $\tau_1, \ldots, \tau_k$ is $S - e_{2n+2k-1}$-regular.

This is the end of the **Long List Of Special Subcases** given by subcases of case III.

This completes our analysis of the summands arising from the expansion of $\Psi_1(S \otimes \phi)$.

Summands arising from Ψ_2. Next, we look at $\Psi_2(S \otimes \phi)$. Let $j \in \{0, \ldots, N\}$ and let $\phi_i : \Delta^{n_i} \to X$ for each $i \in \{1, \ldots, k\}$ be such that $\partial_j \phi = [S; \phi_1, \ldots, \phi_k]$. Suppose $j < N$. Suppose v_{j+1} is associated to the edges $e_{2s-1}, \ldots, e_{2t-1}$. Then let $i = h(e_{2s-1})$ and let l be such that v_{j+1} is the l-th vertex of σ_i. If $j = N$ then suppose v_{N-1} is associated to the edges $e_{2s-1}, \ldots, e_{2t-1}$. Then let $i = h_t$ and let $l = n_i + 1$. Let $\bar{\phi}_i : \Delta^{n_i+1} \to X$ be the map such that:

(i) $\bar{\phi}_i = \phi \circ f$ where f is the inclusion of a face,

(ii) $\bar{\phi}_i|_{\delta_l \Delta^{n_i+1}} = \phi_i$ and

(iii) $\bar{\phi}_i(u_l) = \phi(v_j)$ where Δ^{n_i+1} is spanned by $u_0, \ldots, u_{n_i+1}$.

Let $\bar{\phi}_\gamma = \phi_\gamma$ for $\gamma \neq i$. Then $\bar{\phi}_1, \ldots, \bar{\phi}_k$ is S-regular so $\bar{\phi}_1 \otimes \cdots \otimes \bar{\phi}_{i-1} \otimes \partial_l \bar{\phi}_i \otimes \bar{\phi}_{i+1} \otimes \cdots \otimes \bar{\phi}_k$ is a summand of $\Psi_1(S \otimes \phi)$. Further

$$\bar{\phi}_1 \otimes \cdots \otimes \bar{\phi}_{i-1} \otimes \partial_l \bar{\phi}_i \otimes \bar{\phi}_{i+1} \otimes \cdots \otimes \bar{\phi}_k = \phi_1 \otimes \cdots \otimes \phi_k$$

and these two summands were paired off in case II above.

Summands arising from Ψ_3. It remains to examine the summands arising form the expansion of $\Psi_3(S \otimes \phi)$. Let $e_{2a-1} \in D_S$. Let $\phi_1, \ldots, \phi_k$ be such that $\phi = [S - e_{2a-1}, \phi_1, \ldots, \phi_k]$. Note that if $h(e_{2a-3}) = h_a = h_{a+1}$ then no collection $\phi_1, \ldots, \phi_k$ would be either S-regular or $S - e$-regular for any $e \in D_S$ so $(\Psi_1 + \Psi_2 + \Psi_3)(S \otimes \phi)$ would be zero automatically. We have five remaining cases:

1. $h_{a-1} = h_a$ and either $h_a \neq h_{a+1}$ or $a = n + k$

2. $h_{a+1} = h_a$ and either $h_a \neq h_{a-1}$ or $a = 1$

3. $h_{a-1} \neq h_a \neq h_{a+1}$ and $1 < a < n - k$

4. $h_1 \neq h_2$ and $a = 1$

5. $h_{n+k-1} \neq h_{n+k}$ and $a = n + k$.

Case 1. In this case, $\phi_1, \ldots, \phi_k$ is both $S - e_{2a-1}$-regular and $S - e_{2a-3}$-regular and $[S - e_{2a-1}, \phi_1, \ldots, \phi_k] = [S - e_{2a-3}, \phi_1, \ldots, \phi_k] = \phi$. So we can pair off the two summands in the expansion of $\Psi_3(S \otimes \phi)$ arising from the facts that $\phi_1, \ldots, \phi_k$ is $S - e_{2a-1}$-regular and $\phi_1, \ldots, \phi_k$ is $S - e_{2a-3}$-regular.

Case 2. In this case, $\phi_1, \ldots, \phi_k$ is both $S - e_{2a-1}$-regular and $S - e_{2a+1}$-regular and $[S - e_{2a-1}, \phi_1, \ldots, \phi_k] = [S - e_{2a+1}, \phi_1, \ldots, \phi_k] = \phi$. So we can pair off the two summands in the expansion of $Psi_3(S \otimes \phi)$ arising from the facts that $\phi_1, \ldots, \phi_k$ is $S - e_{2a-1}$-regular and $\phi_1, \ldots, \phi_k$ is $S - e_{2a+1}$-regular.

Case 3. Note that for each i, $\phi_i = \phi \circ f_i$, where f_i is the inclusion of a face. Let σ_i be the image of f_i. Let $v_0, \ldots, v_N$ be the vertices of Δ^N. Let $V_0, \ldots, V_{n+k}$ be the pairwise common vertices of $\sigma_1, \ldots, \sigma_k$. Let $i = h_a$. Let $u_0, \ldots, u_{n_i}$ be the vertices of σ_i. Since $e_{2a-1} \in D_S$, there is a $c \in \{1, \ldots, n+k\}$ such that $c \neq a$ and $h(e_{2c-1}) = i$. If every such c is greater than a then let

$j = 0$. If every such c is less than a then let $j = n_i + 1$. If there are $c_1, c_2 \in \{1, \ldots, n+k\}$ such that $c_1 < a < c_2$ and $h_{c_1} = h_{c_2} = i$ then:

(i) let $c \in \{1, \ldots, n+k\}$ be the lowest such that $c > a$ and $h_c = a$ and

(ii) let $j \in \{1, \ldots, n_i\}$ be the lowest such that e_{2c-1} is associated to u_j.

There are two cases to consider:

3.a. $V_{a-1} \notin \{u_0, \ldots, u_{n_i}\}$.

3.b. $V_{a-1} \in \{u_0, \ldots, u_{n_i}\}$.

Case 3.a. Let $\bar{\phi}_i : \Delta^{n_i+1} \to X$ be the map such that:

(i) $\bar{\phi}_i = \phi \circ f$ where f is the inclusion of a face,

(ii) $\bar{\phi}_i|_{\delta_j \Delta^{n_i+1}} = \phi_i$ and

(iii) $\bar{\phi}_i$ takes the j-th vertex of Δ^{n_i+1} to V_{a-1}.

Let $\bar{\phi}_\gamma = \phi_\gamma$ for $\gamma \neq i$. Then $\bar{\phi}_\gamma = \phi \circ f_\gamma$ for each γ (including $\gamma = i$). Let $\tau_\gamma = im(f_\gamma)$. Then $\tau_1, \ldots, \tau_k$ is S-regular so $\bar{\phi}_1, \ldots, \bar{\phi}_k$ is S-regular and

$$\bar{\phi}_1 \otimes \cdots \otimes \bar{\phi}_{i-1} \otimes \partial_j \bar{\phi}_i \otimes \bar{\phi}_{i+1} \otimes \cdots \otimes \bar{\phi}_k = \phi_1 \otimes \cdots \otimes \phi_k$$

so that $\phi_1 \otimes \cdots \otimes \phi_k$ is equal to a summand of $\Psi(S \otimes \phi)$. These two summands were paired off in case III.1 above.

Example. Suppose $k = 3$, $N = 4$ and $S = < h_1, \ldots, h_5 >$ is the step diagram in figure 5. Let $a = 2$. Suppose the vertices of $\sigma_1, \sigma_2, \sigma_3$ are those suggested by the following diagram:

σ_3	v_0	v_1				
σ_2				v_3	v_4	
σ_1		v_1	v_2	v_3		

Then $\sigma_1, \sigma_2, \sigma_3$ is $S - e_{2a-1}$-regular. Further, $i = 2$, $n_i = 1$, $u_0 = v_3$, $u_1 = v_4$, $c = 4$ and $V_{a-1} = v_1$. Then $V_{a-1} \notin \{u_0, u_1\}$ so this is an example of case 3.a. The vertices of τ_1, τ_2, τ_3 are those suggested by the following diagram:

τ_3	v_0	v_1				
τ_2		v_1		v_3	v_4	
τ_1		v_1	v_2	v_3		

so that τ_1, τ_2, τ_3 is S-regular.

Case 3.b. This case is another of the **Long List Of Special Subcases** referred to just after the statement of theorem 5.5.1.

Let α be such that $v_\alpha = V_{a-1}$. Then $e_{2s_1-1}, \ldots, e_{2a-3}, e_{2a+1}, \ldots, e_{2s_0-1}$ are associated with v_α for some s_0, s_1 where e_{2s_1-3}, e_{2s_0+1} are not associated with v_α. There are two subcases to consider:

3.b.1. $s_1 + 1 \neq a$

3.b.2. $s_1 + 1 = a$.

Case 3.b.1. Let $i_1 = h_{s_1+1}$. We have five further subcases to consider:

3.b.1(i). $v_{\alpha-1}$ is not a vertex in σ_{i_1}

3.b.1(ii). $v_{\alpha-1}$ is a vertex in σ_{i_1}. Note that $v_{\alpha-1}$ is not associated to e_{2s_1+1}. Let e_{2s_2-1} be the edge of height h_{s_1+1} associated to $v_{\alpha-1}$. Either $\alpha = 1$ or $v_{\alpha-2}$ is not associated to e_{2s_2-1}.

3.b.1(iii). $v_{\alpha-1}$ is a vertex in σ_{i_1}, $\alpha > 1$, $v_{\alpha-2}$ is associated to e_{2s_2-1} and $v_{\alpha-2}$ is not a vertex of $\sigma_{h_{s_1}}$.

3.b.1(iv). For each $t \geq 3$, define σ_{i_t} and s_t inductively as follows. Suppose σ_{i_p}, s_p have been defined for every $p \in \{1, \ldots, t-1\}$. Let $i_t = h_{s_{t-1}}$. Suppose $v_{\alpha-t+1}$ is a vertex of $\sigma_{i_{t-1}}$ and $v_{\alpha-p}$ is associated to e_{2s_p-1} for every $p \in \{1, \ldots, t-1\}$. Let e_{2s_t-1} be the edge associated to $v_{\alpha-t+1}$ of height $h_{s_{t-2}}$. Either $\alpha = t-1$ or $v_{\alpha-t}$ is not associated to

e_{2s_t-1}.

3.b.1(v). Let $t \geq 3$. $v_{\alpha-t+1}$ is a vertex of $\sigma_{i_{t-1}}$, $\alpha > t-1$, $v_{\alpha-p}$ is associated to e_{2s_p-1} for every $p \in \{1, \ldots, t\}$ and $v_{\alpha-t}$ is not a vertex of σ_{i_t}.

Case 3.b.1(i). Let j_1 be such that the j_1-th vertex of σ_{i_1} is v_α. Let $\bar{\phi}_{i_1}$: $\Delta^{n_{i_1}+1} \rightarrow X$ be such that:

(i) $\bar{\phi}_{i_1} = \phi \circ f$ where f is the inclusion of a face

(ii) $\bar{\phi}_{i_1}|_{\delta_{j_1}\Delta^{n_{i_1}+1}} = \phi_{i_1}$

(iii) $\bar{\phi}_{i_1}$ takes the j_1-th vertex of $\Delta^{n_{i_1}+1}$ to $v_{\alpha-1}$

Let $\bar{\phi}_\gamma = \phi_\gamma$ for every $\gamma \neq i_1$. Then $\bar{\phi}_\gamma = \phi \circ \bar{f}_\gamma$ where $\bar{f}_\gamma$ is the inclusion of a face. Let $\tau_1, \ldots, \tau_k$ be the images of $\bar{f}_1, \ldots, \bar{f}_k$ respectively. Then:

(i) $V_{b-1}, \ldots, V_b$ are vertices in τ_{h_b} for every $b \in \{1, \ldots, s_1 - 1\}$

(ii) $V_{s_1-1}, \ldots, v_{\alpha-1}$ are vertices in $\tau_{h_{s_1}}$

(iii) $v_{\alpha-1}, v_\alpha$ are vertices in $\tau_{h_{s_1+1}}$

(iv) v_α is a vertex in τ_{h_b} for every $b \in \{s_1+2, \ldots, s_0 - 1\}$

(v) $V_{b-2}, \ldots, V_{b-1}$ are vertices in τ_{h_b} for every $b \in \{s_0, \ldots, n+k\}$

and all the vertices in each of $\tau_1, \ldots, \tau_k$ have been accounted for.. Hence $\bar{\phi}_1, \ldots, \bar{\phi}_k$ is S-regular and the summand $\bar{\phi}_1 \otimes \cdots \otimes \bar{\phi}_{i_1-1} \otimes \partial_{j_1}\bar{\phi}_{i_1} \otimes \bar{\phi}_{i_1+1} \otimes \cdots \otimes \bar{\phi}_k$ of $\Psi_1(S \otimes \phi)$ is equal to the summand $\phi_1 \otimes \cdots \otimes \phi_k$ of $\Psi_3(S \otimes \phi)$. Further, these two summands were paired off in case III.3.b(i) above.

Example. Suppose $k = 3$, $N = 3$ and $S =< e_1, \ldots, e_9 >$ is the step diagram in figure 5. Let $a = 4$. Suppose the vertices of $\sigma_1, \sigma_2, \sigma_3$ are those suggested by the following diagram:

σ_3 $\quad$ $v_0\ v_1\ v_2\ v_3$

σ_2 $v_1\ v_2$

σ_1 v_2

Then $\sigma_1, \sigma_2, \sigma_3$ is $S - e_{2a-1}$-regular. Further, $i = 2$, $n_i = 1$, $u_0 = v_1$, $u_1 = v_2$, $c = 2$ and $V_{a-1} = v_2$. Then $V_{a-1} = u_1$ so this is an example of case 3.b. Further $s_1 + 1 = 1 \neq a$ so this is an example of case 3.b.1. Further $v_{\alpha-1} = v_1$ is not a vertex in $\sigma_{i_1} = \sigma_1$ so this is an example of case 3.b.1(i). The vertices of τ_1, τ_2, τ_3 are those suggested by the following diagram:

τ_3 $v_0\ v_1\ v_2\ v_3$

τ_2 $v_1\ v_2$

τ_1 $v_1\ v_2$

so that τ_1, τ_2, τ_3 is S-regular.

Case 3.b.1(ii) In this case:

(i) $V_{b-1}, \ldots, V_b$ are vertices in τ_{h_b} for every $b \in \{1, \ldots, s_2 - 2\}$

(ii) If e_{2s_2-3} exists then $V_{s_2-2}, \ldots, v_{alpha-1}$ are vertices in $\tau_{h_{s_2-1}}$

(iii) $v_{\alpha-1}$ is a vertex in τ_{h_b} for every $b \in \{s_2 + 1, \ldots, s_1\}$

(iv) $v_{\alpha-1}, v_\alpha$ are vertices in $\tau_{h_{s_1+1}}$

(v) v_α is a vertex in τ_{h_b} for every $b \in \{s_1 + 2, \ldots, s_0 - 1\}$

(vi) $v_\alpha, \ldots, V_{s_0+1}$ are vertices in $\tau_{h_{s_0}}$

(vii) $V_{b-1}, \ldots, V_b$ are vertices in τ_{h_b} for every $b \in \{s_0 + 1, \ldots, n + k\}$

and all the vertices in each of $\tau_1, \ldots, \tau_k$ have been accounted for.. Hence $\phi_1, \ldots, \phi_k$ is both $S - e_{2a-1}$-regular and $S - e_{2s_2-1}$-regular. Therefore we pair off the summand of $\Psi_3(S \otimes \phi)$ arising from the fact that $\phi_1, \ldots, \phi_k$ is $S - e_{2a-1}$-regular with the summand of $\Psi_3(S \otimes \phi)$ arising from the fact that $\phi_1, \ldots, \phi_k$ is $S - e_{2s_2-1}$-regular.

Example. Suppose $k = 3$, $N = 3$ and $S =< h_1, \ldots, h_6 >$ is the step diagram

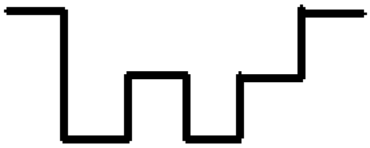


figure 9

in figure 9. Let $a = 5$. Suppose the vertices of $\sigma_1, \sigma_2, \sigma_3$ are those suggested by the following diagram:

σ_3	$v_0\ v_1\ v_2\ v_3$
σ_2	$v_1\ v_2$
σ_1	$v_1\ v_2$

Then $\sigma_1, \sigma_2, \sigma_3$ is $S - e_{2a-1}$-regular. Further, $i = 2$, $n_i = 1$, $u_0 = v_1$, $u_1 = v_2$, $c = 3$ and $V_{a-1} = v_2$. Then $V_{a-1} = u_1$ so this is an example of case 3.b. Further $s_1 + 1 = 4 \neq a$ so this is an example of case 3.b.1. Further $v_{\alpha-1} = v_1$ is a vertex in $\sigma_{i_1} = \sigma_1$ and $v_{\alpha-2} = v_0$ is not associated to $e_{2s_2-1} = e_3$ so this is an example of case 3.b.1(ii). Notice that $\sigma_1, \sigma_2, \sigma_3$ is $S - e_3$-regular.

Case 3.b.1(iii). Let $i_2 = h_{s_1}$. Let j_2 be such that the j_2-th vertex of σ_{i_2} is $v_{\alpha-1}$. Let $\bar{\phi}_{i_2} : \Delta^{n_{i_2}+1} \to X$ be such that:

(i) $\bar{\phi}_{i_2} = \phi \circ f$ where f is the inclusion of a face

(ii) $\bar{\phi}_{i_2}|_{\delta_{j_2}\Delta^{n_{i_2}+1}} = \phi_{i_2}$

(iii) $\bar{\phi}_{i_2}$ takes the j_2-th vertex of $\Delta^{n_{i_2}+1}$ to $v_{\alpha-2}$

Let $\bar{\phi}_\gamma = \phi_\gamma$ for every $\gamma \neq i_2$. Then $\bar{\phi}_\gamma = \phi \circ \bar{f}_\gamma$ where $\bar{f}_\gamma$ is the inclusion of a face. Let $\tau_1, \ldots, \tau_k$ be the images of $\bar{f}_1, \ldots, \bar{f}_k$ respectively. Then:

(i) $V_{b-1}, \ldots, V_b$ are vertices in τ_{h_b} for every $b \in \{1, \ldots, s_2 - 1\}$

(ii) $V_{s_2-1}, \ldots, v_{\alpha-2}$ are vertices in $\tau_{h_{s_2}}$

(iii) $v_{\alpha-2}$ is a vertex in τ_{h_b} for every $b \in \{s_2 + 1, \ldots, s_1 - 1\}$

(iv) $v_{\alpha-2}, v_{\alpha-1}$ are vertices in $\tau_{h_{s_1}}$

(v) $v_{\alpha-1}, v_\alpha$ are vertices in $\tau_{h_{s_1+1}}$

(vi) v_α is a vertex in τ_{h_b} for each $b \in \{s_1+1, \ldots, a\}$

(vii) $v_\alpha, \ldots, V_{a+1}$ are vertices in $\tau_{h_{a+1}}$

(viii) $V_{b-2}, \ldots, V_{b-1}$ are vertices in τ_{h_b} for every $b \in \{a+2, \ldots, n+k\}$

and all the vertices in each of $\tau_1, \ldots, \tau_k$ have been accounted for.. Hence $\bar{\phi}_1, \ldots, \bar{\phi}_k$ is S-regular and the summand $\bar{\phi}_1 \otimes \cdots \otimes \bar{\phi}_{i_1-1} \otimes \partial_{j_1}\bar{\phi}_{i_1} \otimes \bar{\phi}_{i_1+1} \otimes \cdots \otimes \bar{\phi}_k$ of $\Psi_1(S \otimes \phi)$ is equal to the summand $\phi_1 \otimes \cdots \otimes \phi_k$ of $\Psi_3(S \otimes \phi)$. Further, these two summands were paired off in case III.3.b(v') above.

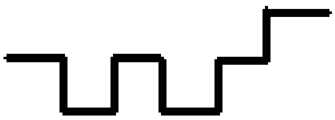


figure 10

Example. Suppose $k = 3$, $N = 4$ and $S =< h_1, \ldots, h_6 >$ is the step diagram in figure 10. Let $a = 5$. Suppose the vertices of $\sigma_1, \sigma_2, \sigma_3$ are those suggested by the following diagram:

σ_3 $\quad\quad\quad\quad\quad\quad\quad$ $v_3\ v_4$

σ_2 $\quad\quad$ $v_0 \quad\ v_2\ v_3$

σ_1 $\quad\quad$ $v_0\ v_1\ v_2\ v_3$

Then $\sigma_1, \sigma_2, \sigma_3$ is $S - e_{2a-1}$-regular. Further, $i = 2$, $n_i = 3$, $u_0 = v_0$, $u_1 = v_1$, $u_2 = v_2$, $u_3 = v_3$, $c = 3$ and $V_{a-1} = v_3$. Then $V_{a-1} = u_3$ so this is an example of case 3.b. Further $s_1+1 = 1 \neq a$ so this is an example of case 3.b.1. Further $v_{\alpha-1} = v_2$ is a vertex in $\sigma_{i_1} = \sigma_1$, $v_\alpha - 2 = v_1$ is associated to $e_{2s_2-1} = e_3$ and $v_\alpha - 2 = v_1$ is not a vertex in $\sigma_{h_{s_1}} = \sigma_2$ so this is an example of case 3.b.1(iii).

The vertices of τ_1, τ_2, τ_3 are those suggested by the following diagram:

τ_3				v_3	v_4
τ_2	v_0	v_1	v_2	v_3	
τ_1	v_0	v_1	v_2	v_3	

so that τ_1, τ_2, τ_3 is S-regular.

Case 3.b.1(iv). In this case:

(i) $V_{b-1}, \dots, V_b$ are vertices in τ_{h_b} for every $b \in \{1, \dots, s_t - 2\}$

(ii) If e_{2s_t-3} exists then $V_{s_t-2}, \dots, v_{\alpha-t+1}$ are vertices in $\tau_{h_{s_t-1}}$

(iii) $v_{\alpha-p}$ is a vertex in τ_{h_b} for every $b \in \{s_{p+1}+1, \dots, s_p\}$ for every $p \in \{1, \dots, t-1\}$

(iv) $v_{\alpha-p}, v_{\alpha-p+1}$ are vertices in $\tau_{h_{s_p+1}}$ for every $p \in \{1, \dots, t-1\}$

(v) v_α is a vertex in τ_{h_b} for every $b \in \{s_1+1, \dots, s_0-1\}$

(vi) $v_\alpha, \dots, V_{s_0}$ are vertices in $\tau_{h_{s_0}}$

(vii) $V_{b-1}, \dots, V_b$ are vertices in τ_{h_b} for every $b \in \{s_0+1, \dots, n+k\}$

and all the vertices in each of $\tau_1, \dots, \tau_k$ have been accounted for.. Hence $\phi_1, \dots, \phi_k$ is both $S - e_{2a-1}$-regular and $S - e_{2s_t-1}$-regular. Therefore we pair off the summand of $\Psi_3(S \otimes \phi)$ arising from the fact that $\phi_1, \dots, \phi_k$ is $S - e_{2a-1}$-regular with the summand of $\Psi_3(S \otimes \phi)$ arising from the fact that $\phi_1, \dots, \phi_k$ is $S - e_{2s_t-1}$-regular.

figure 11

Example. Suppose $k = 4$, $N = 3$ and S is the step diagram in figure 11.

Let $a = 6$. Suppose the vertices of $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ are those suggested by the following diagram:

σ_4				v_3	v_4
σ_3		v_1	v_2	v_3	
σ_2		v_1	v_2	v_3	
σ_1	v_0	v_1	v_2		

Then $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ is both $S - e_{2a-1}$-regular and $S - e_{2s_3-1}$-regular. This is an example of case 3.b.1(iv) with $t = 3$.

Case 3.b.1(v). Let j_t be such that the j_t-th vertex of σ_{i_t} is $v_{\alpha-t+1}$. Construct $\bar{\phi}_1, \ldots, \bar{\phi}_k$ so that $\bar{\phi}_\gamma = \phi_\gamma$ for every $\gamma \neq i_t$ and

(i) $\bar{\phi}_{i_t} = \phi \circ f$ where f is the inclusion of a face

(ii) $\phi_{i_t}|_{\delta_{j_t}\Delta^{n_{i_t}+1}} = \phi_{i_t}$

(iii) $\bar{\phi}_{i_t}$ takes the j_t-th vertex of $\Delta^{n_{i_t}+1}$ to $v_{\alpha-t+1}$

Then:

(i) $V_{b-1}, \ldots, V_b$ are vertices in τ_{h_b} for every $b \in \{1, \ldots, s_t - 1\}$

(ii) $V_{s_t-1}, \ldots, v_{\alpha-t}$ are vertices in $\tau_{h_{s_t}}$

(iii) $v_{\alpha-p}$ is a vertex in τ_{h_b} for every $b \in \{s_p + 1, \ldots, s_{p-1} - 1\}$ for every $p \in \{2, \ldots, t\}$

(iv) $v_{\alpha-p-1}, v_{\alpha-p}$ are vertices in $\tau_{h_{s_p}}$ for every $p \in \{1, \ldots, t-1\}$

(v) $v_{\alpha-1}, v_\alpha$ are vertices in $\tau_{h_{s_1+1}}$

(vi) v_α is a vertex in τ_{h_b} for every $b \in \{s_1 + 2, \ldots, a\}$

(vii) $v_\alpha, \ldots, V_{a+1}$ are vertices in $\tau_{h_{a+1}}$

(viii) $V_{b-1}, \ldots, V_b$ are vertices in τ_{h_b} for every $b \in \{a + 2, \ldots, n + k\}$

and all the vertices in each of $\tau_1, \ldots, \tau_k$ have been accounted for.. Hence $\bar{\phi}_1, \ldots, \bar{\phi}_k$ is S-regular and the summand $\bar{\phi}_1 \otimes \cdots \otimes \bar{\phi}_{i_1-1} \otimes \partial_{j_1}\bar{\phi}_{i_1} \otimes \bar{\phi}_{i_1+1} \otimes \cdots \otimes \bar{\phi}_k$ of $\Psi_1(S \otimes \phi)$ is equal to the summand $\phi_1 \otimes \cdots \otimes \phi_k$ of $\Psi_3(S \otimes \phi)$.

Further, these two summands were paired off in case III.3.b(v') above.

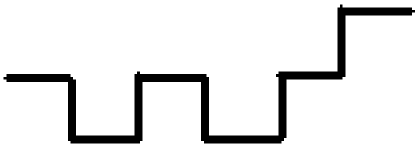


figure 12

Example. Suppose $k = 3$, $N = 4$ and S is the step diagram in figure 12. Let $a = 5$. Suppose the vertices of $\sigma_1, \sigma_2, \sigma_3$ are those suggested by the following diagram:

σ_3 $\qquad\qquad\qquad\qquad$ $v_3\ v_4$

σ_2 $\qquad\qquad$ $v_0\ v_1\ v_2\ v_3$

σ_1 $\qquad\qquad\qquad$ $v_1\ v_2\ v_3$

Then $\sigma_1, \sigma_2, \sigma_3$ is $S - e_{2a-1}$-regular. Further, this is an example of case 3.b.1(v) with $t = 3$. The vertices of τ_1, τ_2, τ_3 are those suggested by the following diagram:

τ_3 $\qquad\qquad\qquad\qquad$ $v_3\ v_4$

τ_2 $\qquad\qquad$ $v_0\ v_1\ v_2\ v_3$

τ_1 $\qquad\qquad$ $v_0\ v_1\ v_2\ v_3$

so that τ_1, τ_2, τ_3 is S-regular.

Case 3.b.2. Suppose first that $s_0 - 1 = a$. Then $s_1 = s_0 - 2$ so either $c = s_0$ or $c = s_1$. In the former case, $h_{a+1} = h(e_{2s_0-1}) = h(e_{2c-1}) = h_a$ and in the latter case, $h(e_{2a-3}) = h_{s_1} = h(e_{2c-1}) = h_a$. Either way, this contradicts the hypothesis of case 3. Therefore we can assume that $s_0 - 1 \neq a$.

Since the case 3.b.2 is the 'mirror image' of case 3.b.1, no examples will be given since the reader can utilise the examples from case 3.b.1 by reflecting all diagrams in a vertical line.

Let $i_1 = h(e_{2s_0-3})$. For case 3.2.b, let $r_1 = s_0$. We have five further subcases to consider:

3.b.2(i). $v_{\alpha+1}$ is not a vertex in σ_{i_1}

3.b.2(ii). $v_{\alpha+1}$ is a vertex in σ_{i_1}. Note that $v_{\alpha+1}$ is not associated to e_{2r_1-3}. Let e_{2r_2-1} be the edge of height h_{r_1-1} associated to $v_{\alpha+1}$. Either $\alpha = N-1$ or $v_{\alpha+2}$ is not associated to e_{2r_2-1}.

3.b.2(iii). $v_{\alpha+1}$ is a vertex in σ_{i_1}, $\alpha < N-1$, $v_{\alpha+2}$ is associated to e_{2r_2-1} and

$v_{\alpha+2}$ is not a vertex in $\sigma_{h_{r_1}}$.

3.b.2(iv). For each $t \geq 3$, define σ_{i_t} and s_t inductively as follows. Suppose

σ_{i_p}, s_p have been defined for every $p \in \{1, \ldots, t-1\}$. Let $i_t = h_{r_{t-1}}$. Suppose $v_{\alpha+t-1}$ is a vertex of $\sigma_{i_{t-1}}$ and $v_{\alpha+p}$ is associated to e_{2r_p-1} for every $p \in \{1, \ldots, t-1\}$. Let e_{2r_t-1} be the edge associated to $v_{\alpha+t-1}$ of height $h_{r_{t-2}}$. Either $\alpha = t-1$ or $v_{\alpha+t}$ is not associated to e_{2r_t-1}.

3.b.2(v). Let $t \geq 3$. $v_{\alpha+t-1}$ is a vertex of $\sigma_{i_{t-1}}$, $\alpha < N-t+1$, $v_{\alpha-p}$ is associated to e_{2r_p-1} for every $p \in \{1, \ldots, t\}$ and $v_{\alpha+t}$ is not a vertex of σ_{i_t}.

Case 3.b.2(i). Let j_1 be such that the j_1-th vertex of σ_{i_1} is v_α. Let $\bar{\phi}_{i_1} : \Delta^{n_{i_1}+1} \to X$ be such that:

(i) $\bar{\phi}_{i_1} = \phi \circ f$ where f is the inclusion of a face

(ii) $\bar{\phi}_{i_1}|_{\delta_{j_1}\Delta^{n_{i_1}+1}} = \phi_{i_1}$

(iii) $\bar{\phi}_{i_1}$ takes the j_1-th vertex of $\Delta^{n_{i_1}+1}$ to $v_{\alpha+1}$

Let $\bar{\phi}_\gamma = \phi_\gamma$ for every $\gamma \neq i_1$. Then $\bar{\phi}_\gamma = \phi \circ \bar{f}_\gamma$ where $\bar{f}_\gamma$ is the inclusion of a face. Let $\tau_1, \ldots, \tau_k$ be the images of $\bar{f}_1, \ldots, \bar{f}_k$ respectively. Then:

(i) $V_{b-1}, \ldots, V_b$ are vertices in τ_{h_b} for every $b \in \{1, \ldots, s_1 - 1\}$

(ii) $V_{s_1-1}, \ldots, v_{\alpha-1}$ are vertices in $\tau_{h_{s_1}}$

(iii) $v_{\alpha-1}$ is a vertex in $\tau_{h_{b+1}}$ for every $b \in \{s_1 + 1, \ldots, r_1 - 2\}$

(iv) $v_{\alpha-1}, v_\alpha$ is a vertex in $\tau_{h_{r_1-1}}$

(v) $v_\alpha, \ldots, V_{r_1}$ are vertices in $\tau_{h_{r_1}}$

(vi) $V_{b-1}, \ldots, V_b$ are vertices in τ_{h_b} for every $b \in \{r_1 + 1, \ldots, n + k\}$

and all the vertices in each of $\tau_1, \ldots, \tau_k$ have been accounted for.. Hence $\bar{\phi}_1, \ldots, \bar{\phi}_k$ is S-regular and the summand $\bar{\phi}_1 \otimes \cdots \otimes \bar{\phi}_{i_1-1} \otimes \partial_{j_1}\bar{\phi}_{i_1} \otimes \bar{\phi}_{i_1+1} \otimes \cdots \otimes \bar{\phi}_k$ of $\Psi_1(S \otimes \phi)$ is equal to the summand $\phi_1 \otimes \cdots \otimes \phi_k$ of $\Psi_3(S \otimes \phi)$. Further, these two summands were paired off in case III.2.b(i) above.

Case 3.b.2(ii) In this case:

(i) $V_{b-1}, \ldots, V_b$ are vertices in τ_{h_b} for every $b \in \{1, \ldots, s_1 - 1\}$

(ii) $V_{s_1+1}, \ldots, v_\alpha$ are vertices in $\tau_{h_{s_1}}$

(iii) v_α is a vertex in τ_{h_b} for every $b \in \{s_1 + 1, \ldots, r_1 - 2\}$

(iv) $v_{\alpha-1}, v_\alpha$ are vertices in $\tau_{h_{r_1-1}}$

(v) $v_{\alpha+1}$ is a vertex in τ_{h_b} for every $b \in \{r_1, \ldots, r_2 - 1\}$

(vi) If e_{2r_2+1} exists then $v_{alpha+1}, \ldots, V_{r_2+1}$ are vertices in $\tau_{h_{r_2+1}}$

(vii) $V_{b-1}, \ldots, V_b$ are vertices in τ_{h_b} for every $b \in \{r_2 + 2, \ldots, n + k\}$

and all the vertices in each of $\tau_1, \ldots, \tau_k$ have been accounted for.. Hence $\phi_1, \ldots, \phi_k$ is both $S - e_{2a-1}$-regular and $S - e_{2r_2-1}$-regular. Therefore we pair off the summand of $\Psi_3(S \otimes \phi)$ arising from the fact that $\phi_1, \ldots, \phi_k$ is

$S-e_{2a-1}$-regular with the summand of $\Psi_3(S\otimes\phi)$ arising from the fact that $\phi_1,\ldots,\phi_k$ is $S-e_{2r_2-1}$-regular.

Case 3.b.2(iii). Let $i_2=h_{r_1}$. Let j_2 be such that the j_2-th vertex of σ_{i_2} is $v_{\alpha+1}$. Let $\bar\phi_{i_2}:\Delta^{n_{i_2}+1}\to X$ be such that:

(i) $\bar\phi_{i_2}=\phi\circ f$ where f is the inclusion of a face

(ii) $\bar\phi_{i_2}|_{\delta_{j_2}\Delta^{n_{i_2}+1}}=\phi_{i_2}$

(iii) $\bar\phi_{i_2}$ takes the j_2-th vertex of $\Delta^{n_{i_2}+1}$ to $v_{\alpha+2}$

Let $\bar\phi_\gamma=\phi_\gamma$ for every $\gamma\neq i_2$. Then $\bar\phi_\gamma=\phi\circ\bar f_\gamma$ where $\bar f_\gamma$ is the inclusion of a face. Let $\tau_1,\ldots,\tau_k$ be the images of $\bar f_1,\ldots,\bar f_k$ respectively. Then:

(i) $V_{b-1},\ldots,V_b$ are vertices in τ_{h_b} for every $b\in\{1,\ldots,a-2\}$

(ii) $V_{a-2},\ldots,v_\alpha$ are vertices in $\tau_{h_{a-1}}$

(iii) v_α is a vertex in τ_{h_b} for every $b\in\{a,\ldots,r_1-2\}$

(iv) $v_\alpha,v_{\alpha+1}$ are vertices in $\tau_{h_{r_1-1}}$

(v) $v_{\alpha+1},v_{\alpha+2}$ are vertices in $\tau_{h_{r_1}}$

(vi) $v_{\alpha+2}$ is a vertex in τ_{h_b} for each $b\in\{r_1+1,\ldots,r_2-1\}$

(vii) $v_{\alpha+2},\ldots,V_{r_2}$ are vertices in $\tau_{h_{r_2+1}}$

(viii) $V_{b-1},\ldots,V_b$ are vertices in τ_{h_b} for every $b\in\{r_2+1,\ldots,n+k\}$

and all the vertices in each of $\tau_1,\ldots,\tau_k$ have been accounted for.. Hence $\bar\phi_1,\ldots,\bar\phi_k$ is S-regular and the summand $\bar\phi_1\otimes\cdots\otimes\bar\phi_{i_1-1}\otimes\partial_{j_1}\bar\phi_{i_1}\otimes\bar\phi_{i_1+1}\otimes\cdots\otimes\bar\phi_k$ of $\Psi_1(S\otimes\phi)$ is equal to the summand $\phi_1\otimes\cdots\otimes\phi_k$ of $\Psi_3(S\otimes\phi)$. Further, these two summands were paired off in case III.2.b(v') above.

Case 3.b.2(iv). In this case:

(i) $V_{b-1},\ldots,V_b$ are vertices in τ_{h_b} for every $b\in\{1,\ldots,s_1-1\}$

(ii) $V_{s_1-1},\ldots,v_\alpha$ are vertices in $\tau_{h_{s_1}}$

(iii) v_α is a vertex in τ_{h_b} for every $b\in\{s_1+1,\ldots,r_1-1\}$

(iv) $v_{\alpha+p-1},v_{\alpha+p}$ are vertices in $\tau_{h_{r_p+1}}$ for every $p\in\{1,\ldots,t-1\}$

(v) $v_{\alpha+p}$ is a vertex in τ_{h_b} for every $b \in \{r_p+1, \ldots, r_{p+1}\}$ for every $p \in \{1, \ldots, t-1\}$

(vi) If e_{2r_t+1} exists then $v_{\alpha+t-1}, \ldots, V_{r_t+1}$ are vertices in $\tau_{h_{r_t+1}}$

(vii) $V_{b-1}, \ldots, V_b$ are vertices in τ_{h_b} for every $b \in \{r_t+2, \ldots, n+k\}$

and all the vertices in each of $\tau_1, \ldots, \tau_k$ have been accounted for.. Hence $\phi_1, \ldots, \phi_k$ is both $S - e_{2a-1}$-regular and $S - e_{2r_t-1}$-regular. Therefore we pair off the summand of $\Psi_3(S \otimes \phi)$ arising from the fact that $\phi_1, \ldots, \phi_k$ is $S - e_{2a-1}$-regular with the summand of $\Psi_3(S \otimes \phi)$ arising from the fact that $\phi_1, \ldots, \phi_k$ is $S - e_{2s_t-1}$-regular.

Case 3.b.2(v). Let j_t be such that the j_t-th vertex of σ_{i_t} is $v_{\alpha+t-1}$. Construct $\bar{\phi}_1, \ldots, \bar{\phi}_k$ so that $\bar{\phi}_\gamma = \phi_\gamma$ for every $\gamma \neq i_t$ and

(i) $\bar{\phi}_{i_t} = \phi \circ f$ where f is the inclusion of a face

(ii) $\phi_{i_t}|_{\delta_{j_t}\Delta^{n_{i_t}+1}} = \phi_{i_t}$

(iii) $\bar{\phi}_{i_t}$ takes the j_t-th vertex of $\Delta^{n_{i_t}+1}$ to $v_{\alpha+t-1}$

Then:

(i) $V_{b-1}, \ldots, V_b$ are vertices in τ_{h_b} for every $b \in \{1, \ldots, a-2\}$

(ii) $V_{a-2}, \ldots, v_\alpha$ are vertices in $\tau_{h_{a-1}}$

(iii) v_α is a vertex in τ_{h_b} for every $b \in \{a, \ldots, r_1-2\}$

(iv) $v_\alpha, v_{\alpha+1}$ are vertices in $\tau_{h_{r_1-1}}$

(v) $v_{\alpha+p}, v_{\alpha+p+1}$ are vertices in $\tau_{h_{r_p}}$ for every $p \in \{1, \ldots, t-1\}$

(vi) $v_{\alpha+p}$ is a vertex in τ_{h_b} for every $b \in \{r_{p-1}+1, \ldots, r_p-1\}$ for every $p \in \{2, \ldots, t\}$

(vii) $v_{\alpha+t}, \ldots, V_{r_t+1}$ are vertices in $\tau_{h_{r_t}}$

(viii) $V_{b-1}, \ldots, V_b$ are vertices in τ_{h_b} for every $b \in \{r_t+1, \ldots, n+k\}$

and all the vertices in each of $\tau_1, \ldots, \tau_k$ have been accounted for. Hence $\bar{\phi}_1, \ldots, \bar{\phi}_k$ is S-regular and the summand $\bar{\phi}_1 \otimes \cdots \otimes \bar{\phi}_{i_1-1} \otimes \partial_{j_1}\bar{\phi}_{i_1} \otimes \bar{\phi}_{i_1+1} \otimes$

$\cdots \otimes \bar{\phi}_k$ of $\Psi_1(S \otimes \phi)$ is equal to the summand $\phi_1 \otimes \cdots \otimes \phi_k$ of $\Psi_3(S \otimes \phi)$. Further, these two summands were paired off in case III.3.b(v') above.

This ends the **Long List Of Special Subcases** given by subcases of case 3.b.

Case 4. Note that for each i, $\phi_i = \phi \circ f_i$, where f_i is the inclusion of a face. Let σ_i be the image of f_i. Let $v_0, \ldots, v_N$ be the vertices of Δ^N. Let $V_0, \ldots, V_{n+k}$ be the pairwise common vertices of $\sigma_1, \ldots, \sigma_k$. Let $i = h_1$. Let $u_0, \ldots, u_{n_i}$ be the vertices of σ_i.

There are two cases to consider:

4.a. $v_0 \neq u_0$.

4.b. $v_0 = u_0$.

Case 4.a. Let $\bar{\phi}_i : \Delta^{n_i+1} \to X$ be the map such that:

(i) $\bar{\phi}_i = \phi \circ f$ where f is the inclusion of a face,

(ii) $\bar{\phi}_i|_{\delta_0 \Delta^{n_i+1}} = \phi_i$ and

(iii) $\bar{\phi}_i$ takes the 0-th vertex of Δ^{n_i+1} to v_0.

Let $\bar{\phi}_\gamma = \phi_\gamma$ for $\gamma \neq i$. then $\bar{\phi}_1, \ldots, \bar{\phi}_k$ is S-regular and

$$\bar{\phi}_1 \otimes \cdots \otimes \bar{\phi}_{i-1} \otimes \partial_j \bar{\phi}_i \otimes \bar{\phi}_{i+1} \otimes \cdots \otimes \bar{\phi}_k = \phi_1 \otimes \cdots \otimes \phi_k$$

so that $\phi_1 \otimes \cdots \otimes \phi_k$ is equal to a summand of $\Psi(S \otimes \phi)$. These two summands were paired off in case III.2.c above.

Case 4.b. Let $e_3, \ldots, e_{2s_0-1}$ be the edges associated with v_0 for some s_0 where e_{2s_0+1} is not associated with v_0. If $s_0 = 2$ then none of the edges e_{2b-1} for $b > 2$ are associated with v_0. In particular, none of the edges of height i are associated with v_0 so $u_0 \neq v_0$, contradicting the assumption of case 4.b. Therefore we can assume that $s_0 \neq 2$. The rest of case 4.b is virtually identical to case 3.b.2 so a detailed explanation will be omitted.

Case 5. Note that for each i, $\phi_i = \phi \circ f_i$, where f_i is the inclusion of a face. Let σ_i be the image of f_i. Let $v_0, \ldots, v_N$ be the vertices of Δ^N. Let $V_0, \ldots, V_{n+k}$ be the pairwise common vertices of $\sigma_1, \ldots, \sigma_k$. Let $i = h(e_{2n+2k-1})$. Let $u_0, \ldots, u_{n_i}$ be the vertices of σ_i.
There are two cases to consider:

5.a. $v_N \neq u_{n_i}$.

5.b. $v_N = u_{n_i}$.

Case 5.a. Let $\bar{\phi}_i : \Delta^{n_i+1} \to X$ be the map such that:

(i) $\bar{\phi}_i = \phi \circ f$ where f is the inclusion of a face,

(ii) $\bar{\phi}_i|_{\delta_{n_i+1}\Delta^{n_i+1}} = \phi_i$ and

(iii) $\bar{\phi}_i$ takes the $n_i + 1$-th vertex of Δ^{n_i+1} to v_0.

Let $\bar{\phi}_\gamma = \phi_\gamma$ for $\gamma \neq i$. then $\bar{\phi}_1, \ldots, \bar{\phi}_k$ is S-regular and

$$\bar{\phi}_1 \otimes \cdots \otimes \bar{\phi}_{i-1} \otimes \partial_j \bar{\phi}_i \otimes \bar{\phi}_{i+1} \otimes \cdots \otimes \bar{\phi}_k = \phi_1 \otimes \cdots \otimes \phi_k$$

so that $\phi_1 \otimes \cdots \otimes \phi_k$ is equal to a summand of $\Psi(S \otimes \phi)$. These two summands were paired off in case III.3.c above.

Case 5.b. Let $e_{2s_1-1}, \ldots, e_{2n+2k-3}$ be the edges associated with v_N for some s_1 where e_{2s_1-3} is not associated with v_N. If $s_1 = n + k - 1$ then none of the edges e_{2b-1} for $b < n + k - 1$ are associated with v_N. In particular, none of the edges of height i are associated with v_N so $u_{n_i} \neq v_N$, contradicting the assumption of case 5.b. Therefore we can assume that $s_0 \neq 2$. The rest of case 5.b is virtually identical to case 3.b.1 so a detailed explanation will be omitted.

This completes the analysis of all the summands in the expansion of $(\Psi_1 +$

$\Psi_2 + \Psi_3)(S \otimes \phi)$. We conclude that all summands can be paired off with a summand of equal value, so that the total sum is 0. Hence θ is a map of differential graded vector spaces. In order to conclude that $S_*(X)$ is a coalgebra over $\mathcal{S}$, it is necessary to show that (5.5) commutes in the current case. This is clear from the construction and can be checked directly. □

Chapter 6

The step operad with arbitrary coefficients

Let R be a commutative, unital ring. In this chapter, we construct a step operad over R, denoted $\mathcal{S}_R$, with the following properties:

(i) $\mathcal{S}_R$ is an E_∞-operad

(ii) $S_*(X;R)$ is a coalgebra over $\mathcal{S}_R$ for any topological space X

(iii) $\mathcal{S}_{\mathbb{Z}_2} = \mathcal{S}$.

6.1 The definition of $\mathcal{S}_R$

For each integer $k \geq 2, n \geq 0$, let $\mathcal{S}_R(k)_n$ be the free R-module generated by all the step diagrams of height $k-1$ with $k+n$ steps. Let $d : \mathcal{S}_R(k)_n \to \mathcal{S}_R(k)_{n-1}$ be given by

$$dS = \sum_{e_{2i-1} \in D_S} (-1)^{i-1} S - e_{2i-1}$$

where S is a step diagram with edges $e_1, \ldots, e_{2n+2k-1}$. Let $\mathcal{S}_R(1)$ be the free R-module generated by the unique step diagram of height 0 concentrated in degree 0. Then the following lemma can be proved in a similar way to lemma 5.3.1 due to the choice of sign in the formula for d.

Lemma 6.1.1. *$d \circ d : \mathcal{S}_R(k)_n \to \mathcal{S}_R(k)_{n-2}$ is the zero map for all $n \geq 2$.*

Definition 6.1.1. Let $\mathcal{S}_R(k)$ denote the differential graded R-module which is $\mathcal{S}_R(k)_n$ in degree n and has differential d.

The fact that $\mathcal{S}_R(k)$ admits a free action of the symmetric group Σ_k is proved in an identical fashion to lemma 5.3.2. However, we need to take care with signs in the definition of the operad structure map. Let $S_1, \ldots, S_n, T$ be step diagrams such that $T \in \mathcal{S}_R(n)$ and $S_i \in \mathcal{S}_R(k_i)$. Let P be a T-partition of $S_1, \ldots, S_n$. Recall the construction of $S(T; P)$ in section 5.3. The various notations used in this construction will be used in the sequel without further introduction.

For each i, suppose S_i has edges $e^i_1, \ldots, e^i_{2t_i-1}$. Call e^i_{2j-1} *positive* if j is odd and *negative* if j is even. We refer to the assignation of positive or negative as the *polarity* of the step. *Reversing the polarity* of a step means changing it from negative to positive or vice versa. Then the steps of each pre-step diagram $S_{i,j}$ acquire polarities from the polarities of the steps of S_i (in the obvious way). If $S_{i,j} = S^{(s)}_T$ for some *even* s then we reverse the polarities of the step of $S_{i,j}$. (If s is odd then the polarities are preserved.) These new polarities on the steps of $S^{(s)}_T$ induce polarities on the steps of $S(T; P)$ (in the obvious way). Suppose $S(T; P)$ has edges $f_1, \ldots, f_{2t-1}$. Let $m(P) = 0$. Wherever we have a sequence of steps $f_{2p-1}, f_{2p+1}, \ldots, f_{2q-1}$ (with $q > p$) all of which have the same polarity, we increase $m(P)$ by $q - p$.

For example, if $k = 2$ and S_1, S_2, T, P are as in figure 3 then the steps of $S(T; P)$ have the polarities suggested by figure 13 and $m(P) = 2$.

Definition 6.1.2. Suppose $k = k_1 + \cdots + k_n$. Define a map

$$\chi[n, k_1, \ldots, k_n] : \mathcal{S}_R(n) \otimes \mathcal{S}_R(k_1) \otimes \cdots \otimes \mathcal{S}_R(k_n) \to \mathcal{S}_R(k)$$

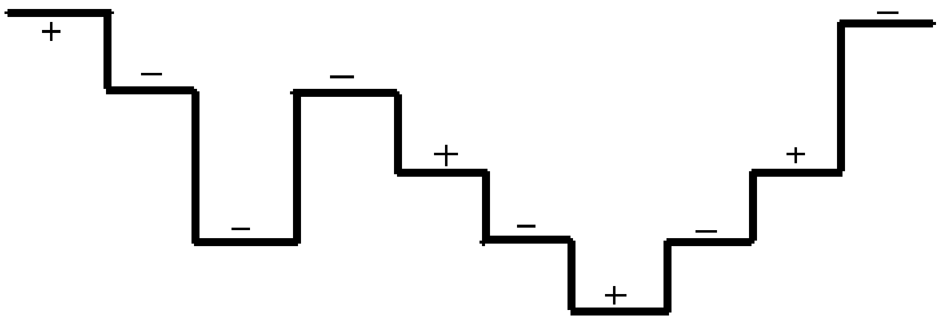


figure 13

$$\chi[n, k_1, \ldots, k_n](T \otimes S_1 \otimes \cdots \otimes S_n) = \sum_{P \in P(T; S_1, \ldots S_n)} (-1)^{m(P)} S(T; P).$$

$\chi[\bullet]$ will be used to denote the sum of $\chi[n, k_1, \ldots, k_n]$ over all possible $n, k_1, \ldots, k_n$. Unless there is cause for confusion, both $\chi[n, k_1, \ldots, k_n]$ and $\chi[\bullet]$ will be denoted χ in the sequel.

The choice of sign in the above definition ensures that χ is a map of differential graded modules (by a similar proof to lemma 5.3.4). Further, the analogue of lemma 5.3.5 holds in this setting. The following is a direct consequence.

Theorem 6.1.2. *The various $\mathcal{S}_R(k)$ fit together to give an operad $\mathcal{S}_R$ with structure map χ.*

Definition 6.1.3. The operad $\mathcal{S}_R$ is called the *step operad over R.*

It will be shown in section 6.3 that $\mathcal{S}_R(k)$ is contractible for every k. It follows that $\mathcal{S}_R$ is an E_∞-operad.

6.2 $S_*(X; R)$ is a coalgebra over $\mathcal{S}_R$

Let X be a topological space. Let $S_*(X)$ denote the singular chains on X with coefficients in R. Let $n \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{Z}_{\geq 1}$. Let $n_1, \ldots, n_k \in \mathbb{Z}_{\geq 0}$. Let

$N = n_1 + \cdots + n_k - n$. Let $\phi_i \in S_{n_i}(X)$ for all $i \in \{1, \ldots, k\}$ be such that ϕ_i is a map $\Delta^{n_i} \to X$ from the standard n_i-simplex to X. Let $S \in \mathcal{S}(k)_n$ be a step diagram. Let $\phi : \Delta^N \to X$. Suppose $\phi = [S; \phi_1, \ldots, \phi_k]$. Then for each i, $\phi_i = \phi \circ f_i$ where f_i is the inclusion of a face.

Let σ_i be the image of f_i. Let $v_0, \ldots, v_N$ be the vertices of Δ^N. Let $V_0, \ldots, V_{n+k}$ be the pairwise-common vertices of $\sigma_1, \ldots, \sigma_k$. Then each σ_i can be expressed as the join

$$\sigma_{i,1} \cdot \cdots \cdot \sigma_{i,t_i}$$

where $\sigma_{i,j}$ is spanned by some set of vertices $v_a, \ldots, v_b$ such that $v_a, v_b \in \{V_0, \ldots, V_{n+k}\}$ and $v_c \notin \{V_0, \ldots, V_{n+k}\}$ whenever $a < c < b$. Let σ'_i be the face of σ_i spanned by all the vertices that are not vertices in σ_j for any $j < i$. Let $\sigma'_{i,j}$ be the intersection of $\sigma_{i,j}$ and σ'_i. Then Δ^N can be expressed uniquely as the join of all the $\sigma'_{i,j}$ listed in a particular order. Let $\pi(\phi_1, \ldots, \phi_n)$ be the permutation of the vertices of Δ^N that takes the ordered set of vertices

$$v_0, \ldots, v_N$$

to the ordered set of vertices given by the join

$$\sigma'_{1,1} \ldots \sigma'_{1,t_1} \cdot \sigma'_{2,1} \ldots \sigma'_{2,t_2} \cdot \cdots \cdot \sigma'_{k,1} \ldots \sigma'_{k,t_k}.$$

Let $|\pi(\phi_1, \ldots, \phi_n)|$ be the sign of the permutation $\pi(\phi_1, \ldots, \phi_n)$.

For each $N, n \in \mathbb{Z}_{\geq 0}$, $k \in \mathbb{Z}_{\geq 1}$, we define a map

$$\theta_{k,N,n} : \mathcal{S}(k)_n \otimes S_N(X) \to \sum_{n_1 + \cdots + n_k = N+n} S_{n_1}(X) \otimes \cdots \otimes S_{n_k}(X)$$

by

$$\theta_{k,N,n}(S \otimes \phi) = \sum_{\phi = [S; \phi_1, \ldots, \phi_k]} (-1)^{|\pi(\phi_1, \ldots, \phi_n)|} \phi_1 \otimes \cdots \otimes \phi_k$$

for each step diagram S and $\phi : \Delta^N \to X$. If it causes no ambiguity, $\theta_{k,N,n}$ will be denoted θ_k or θ. The choice of sign $(-1)^{|\pi(\phi_1,\ldots,\phi_n)|}$ in the above formula allows us to use proof of 5.5.1 with arbitrary coefficients instead of $\mathbb{Z}_2$ to prove the following theorem (compare with Steenrod's choice of sign for the cup-i product in [30]).

Theorem 6.2.1. *$S_*(X)$ is a coalgebra over $\mathcal{S}_R$ with structure map θ.*

6.3 A different construction of $\mathcal{S}_R$

Let X be a finite set. Denote the n-fold Cartesian product copies of X by X^n. Let $\Delta[X]$ be the simplicial set such that the degree m part is

$$\Delta[X]_m = X^{m+1}$$

with face maps given by projections and degeneracies by diagonal maps (in the obvious way). If $X = \{1, \ldots, n\}$ then let $\Delta[X]$ be denoted by $\Delta[n]$. Let $\bar{E}[n]$ be the sub-simplicial set of $\Delta[n]$ generated by $\Delta[n]_m$ for $m < n-1$. Let $E[n]$ be the sub-simplicial set of $\Delta[n]$ such that $E[n]_{k-1}$ consists of all the elements $(x_{\sigma(1)}, \ldots, x_{\sigma(k)})$ for any $(x_1, \ldots, x_k) \in \bar{E}[n]$ and any $\sigma \in \Sigma_k$. Let $NC(\Delta[n])$ denote the normalised chain complex of $\Delta[n]$ with coefficients in R. Let $NC(E[n])$ denote the normalised chain complex of $E[n]$ with coefficients in R. Let

$$\hat{D}[n] = \frac{NC(\Delta[n])}{NC(E[n])}.$$

Theorem 6.3.1. *The homology groups of $\hat{D}[n]$ are $H_{n-1}(\hat{D}[n]) = R$ and $H_m(\hat{D}[n]) = 0$ for $m \neq n-1$.*

Proof. The geometric realisation of $\Delta[n]$ is homotopic to the geometric $n-1$-simplex Δ^{n-1} and the geometric realisation of $E[n]$ is homotopic to the boundary $\partial\Delta^{n-1}$ of Δ^{n-1}. Therefore the relative homology $H_*(\Delta[n], E[n])$ is isomorphic to the relative homology $H_*(\Delta^{n-1}, \partial\Delta^{n-1})$ which is well-known to be R in degree $n-1$ and zero otherwise. Since $H_*(\Delta[n], E[n]) = H_*(\hat{D}[n])$, the result follows. □

Now let $D[n]$ be the $(n-1)$-fold desuspension of $\hat{D}[n]$:

$$D[n] = \Sigma^{1-n}\hat{D}[n].$$

Then $D[n]$ is contractible.

Proposition 6.3.2. *There is an isomorphism of chain complexes $q : \mathcal{S}_R(n) \to D[n]$.*

Proof. Let $S =< h_1, \ldots, h_{n+k} >$ be step diagram of height $n-1$ and $n+k$ steps. Then $(h_1, \ldots, h_{n+k})$ is an element of $\Delta[n]_{n+k-1}$ so represents an element of $D[n]_{n+k-1} = D[n]_k$, which we define to be $q(S)$. A direct calculation shows that q is an isomorphism. □

Corollary 6.3.3. *$\mathcal{S}_R(n)$ is contractible.*

Corollary 6.3.4. *$\mathcal{S}_R(n)$ is an E_∞-operad.*

In fact, the various chain complexes $D[n]$ themselves form an E_∞-operad. Indeed, we can construct structure maps ρ such that the following commutes whenever $k = k_1 + \cdots + k_n$:

$$\begin{array}{ccc} \mathcal{S}_R(n) \otimes \mathcal{S}_R(k_1) \otimes \cdots \otimes \mathcal{S}_R(k_n) & \xrightarrow{\chi} & \mathcal{S}_R(k) \\ \downarrow q & & \downarrow q \\ D[n] \otimes D[k_1] \otimes \cdots \otimes D[k_n] & \xrightarrow{\rho} & D[k] \end{array} \tag{6.1}$$

The structure map ρ is far more simple to describe than χ so we will spend some time looking at it. Firstly, note that $\Delta[n]_m$ can be identified with the set of maps $\{0,\dots,m\} \to \{1,\dots,n\}$. Let $M \in \mathbb{Z}_{\geq 0}$. Suppose $k = k_1 + \cdots + k_n$. Define a map of simplicial sets

$$p : \Delta[n]_M \times \Delta[k_1]_M \times \Delta[k_n]_M \to \Delta[k]_M$$

as follows. Let $(h, f_1, \dots, f_n) \in \Delta[n]_M \times \Delta[k_1]_M \times \cdots \times \Delta[k_n]_M$. Let $\bar{f}_i : \{0,\dots,M\} \to \{k_1+\cdots+k_{i-1}+1, k_1+\cdots+k_{i-1}+2, \dots, k_1+\cdots+k_{i-1}+k_i\}$ be given by $\bar{f}_i(a) = f_i(a) + k_1 + \cdots + k_{i-1}$. Define $p(h, f_1, \dots, f_n)(a) = \bar{f}_{h(a)}(a)$. A direct check shows that this map commutes with faces and degeneracies. p induces a map of (un-normalised) chain complexes

$$C_M(\Delta[n] \times \Delta[k_1] \times \cdots \times \Delta[k_n]) \to C_M(\Delta[k]).$$

Now let $l, l_1, \dots, l_n$ be non-negative integers. Let $N = l + l_1 + \cdots + l_n$. Then the shuffle product defines a map

$$C_{l+n-1}(\Delta[n]) \otimes C_{k_1+l_1-1}(\Delta[k_1]) \otimes \cdots \otimes C_{k_n+l_n-1}(\Delta[k_n]) \to$$

$$C_{k+N-1}(\Delta[n] \times \Delta[k_1] \times \cdots \times \Delta[k_n]).$$

The latter two maps compose to give a map

$$C_{l+n-1}(\Delta[n]) \otimes C_{k_1+l_1-1}(\Delta[k_1]) \otimes \cdots \otimes C_{k_n+l_n-1}(\Delta[k_n]) \to C_{k+N-1}(\Delta[k]).$$

Normalising this map, passing to quotients and desuspending the requisite number of times gives the required map

$$\rho : D_l[n] \otimes D_{l_1}[k_1] \otimes \cdots \otimes D_{l_n}[k_n] \to D_N[k].$$

It is a direct check to see that (6.1) commutes.

6.4 The quadratic suboperad of $\mathcal{S}_R$

We will need the following property of $\mathcal{S}_R$ in the sequel.

Lemma 6.4.1. *The map*

$$\chi[n, k_1, \ldots, k_n] : \mathcal{S}_R(n) \otimes \mathcal{S}_R(k_1) \otimes \cdots \otimes \mathcal{S}_R(k_n) \to \mathcal{S}_R(k)$$

is injective for each $n, k_1, \ldots, k_n \geq 1$.

Proof. Let $l, m_1, \ldots, m_n$ be non-negative integers. Let $T \in \mathcal{S}_R(n)_l$ and $S_i =< h^i_1, \ldots, h^i_{k_i+m_i} >\in \mathcal{S}_R(k_i)_{m_i}$ for each $i \in \{1, \ldots, n\}$. Suppose T has a_i edges of height i for each $i \in \{1, \ldots, n\}$. Consider the T-partition P of $S_1, \ldots, S_n$ given by the collection of a_i integers $k_i + m_i, k_i + m_i, \ldots, k_i + m_i$. Then $S(T; P)$ appears as a summand in $\chi(T \otimes S_1 \otimes \cdots \otimes S_n)$. However, $S(T; P)$ does not appear as a summand in $\chi(T' \otimes S'_1 \otimes \ldots S'_n)$ for any $T' \in \mathcal{S}_R(n)_l$, $S'_1 \in \mathcal{S}_R(k_1)_{m_1}, \ldots, S'_n \in \mathcal{S}_R(k_n)_{m_n}$ such that any of the inequalities $T \neq T'$, $S_1 \neq S'_1, \ldots, S_n \neq S'_n$ hold.

Now let $a \in \mathcal{S}_R(n)_l$, $b_i \in \mathcal{S}_R(k_i)_{m_i}$ for each $i \in \{1, \ldots, n\}$. Then $a = T^1 + \cdots + T^J$ and $b_i = S^1_i + \cdots + S^{j_i}_i$ where T^α, S^γ_β are step diagrams for each α, β, γ. Suppose $a \otimes b_1 \otimes \cdots \otimes b_n \neq 0$. Then

$$\chi(a \otimes b_n \otimes \cdots \otimes b_n) = \sum_{\alpha=1}^{J} \sum_{\beta_1=1}^{j_1} \cdots \sum_{\beta_n=1}^{j_n} \chi(T^\alpha \otimes S^{\beta_1}_1 \otimes \cdots \otimes S^{\beta_n}_n).$$

In the first paragraph of the proof, we constructed a summand in $\chi(T^1 \otimes S^1_1 \otimes \cdots \otimes S^1_n)$ that is not a summand in the expansion of $\chi(T^\alpha \otimes S^{\beta_1}_1 \otimes \cdots \otimes S^{\beta_n}_n)$ whenever any of the inequalities $1 \neq \alpha$, $1 \neq \beta_1, \ldots, 1 \neq \beta_n$ hold. Therefore $\chi(a \otimes b_1 \otimes \cdots \otimes b_n) \neq 0$ so $\chi[n, k_1, \ldots, k_n]$ is injective. □

Note that despite the previous lemma, $\chi[\bullet]$ is not injective. This fact is unnecessary for the rest of the thesis, so will not be proved here. However, it is not difficult to construct a non-trivial element of the kernal of $\chi[\bullet]$.

Definition 6.4.1. The *quadratic suboperad* of $\mathcal{S}_R$ is the operad $\mathcal{Q}_R$ defined as follows. $\mathcal{Q}_R(1) = \mathcal{S}_R(1)$, $\mathcal{Q}_R(2) = \mathcal{S}_R(2)$. If $k > 2$ then suppose $\mathcal{Q}_R(i)$ has been defined for every $i < k$. Then $\mathcal{Q}_R(k)$ is the differential graded vector space generated by the images of the maps

$$\mathcal{Q}_R(n) \otimes \mathcal{Q}_R(k_1) \otimes \cdots \otimes \mathcal{Q}_R(k_n) \hookrightarrow \mathcal{S}_R(n) \otimes \mathcal{S}_R(k_1) \otimes \cdots \otimes \mathcal{S}_R(k_n) \stackrel{\sigma \circ \chi}{\to} \mathcal{S}_R(k)$$

for each $n, k_1, \ldots, k_n \geq 1$ where $k = k_1 + \cdots + k_n$ and each $\sigma \in \Sigma_k$.

The definition ensures that $\mathcal{Q}_R$ is an operad with structure maps χ and ψ induced by those on $\mathcal{S}_R$ of the same name. In addition, the inclusions $\mathcal{Q}_R(k) \hookrightarrow \mathcal{S}_R(k)$ induce a map of operads $\mathcal{Q}_R \to \mathcal{S}_R$ so that any $\mathcal{S}_R$-(co)algebra is also a $\mathcal{Q}_R$-(co)algebra.

Lemma 6.4.2. *$H_m(\mathcal{Q}_R(k)) = 0$ for every $k, m \geq 1$.*

Proof. The cases $k = 1, 2$ are trivial so suppose $k \geq 3$. Suppose, inductively, that $\mathcal{Q}_R(i)$ is contractible for every $i < k$. Let $n \geq 1$. Let $x \in \mathcal{Q}_R(k)_n$ be such that $d(x) = 0$. Then there exist $n, k_1, \ldots, k_n \in \{1, \ldots, k-1\}$, $a \in \mathcal{Q}_R(k)$, $b_i \in \mathcal{Q}_R(k_i)$ for each $i \in \{1, \ldots, n\}$ and $\sigma \in \Sigma_k$ such that $k = k_1 + \cdots + k_n$ and $x = \sigma \circ \chi(a \otimes b_1 \otimes \cdots \otimes b_n)$. Then $d \circ \sigma \circ \chi(a \otimes b_1 \otimes \cdots \otimes b_n) = 0$ so $\sigma \circ \chi \circ d(a \otimes b_1 \otimes \cdots \otimes b_n) = 0$. Since the map

$$\sigma \circ \chi : \mathcal{S}_R(n) \otimes \mathcal{S}_R(k_1) \otimes \cdots \otimes \mathcal{S}_R(k_n) \to \mathcal{S}_R(k)$$

is injective, it follows that $d(a \otimes b_1 \otimes \cdots \otimes b_n) = 0$ so that $d(a) = 0$ and $d(b_i) = 0$ for each i. Note that at least one of $a, b_1, \ldots, b_n$ has degree greater than 0. Suppose, without loss of generality, that the degree of a is greater than 0. Since $\mathcal{Q}_R(n)$ is contractible, it follows that there exists $a' \in \mathcal{Q}_R(n)$ such that $d(a') = a$ so $d(a' \otimes b_1 \otimes \cdots \otimes b_n) = a \otimes b_1 \otimes \cdots \otimes b_n$. Let $x' = \sigma \circ \chi(a' \otimes b_1 \otimes \cdots \otimes b_n)$. Then since $\sigma \circ \chi \circ d(a' \otimes b_1 \otimes \cdots \otimes b_n) = x$, it follows that $d \circ \sigma \circ \chi(a' \otimes b_1 \otimes \cdots \otimes b_n) = x$ so $d(x') = x$. Hence every cycle in $\mathcal{Q}_R(k)_m$ bounds for $m \geq 1$ so $H_m(\mathcal{Q}_R(k)) = 0$. $\square$

Lemma 6.4.3. *$H_0(\mathcal{Q}_R(k)) = R$.*

Proof. This is trivially true for $k = 1, 2$ so suppose $k \geq 3$ and suppose that $H_0(\mathcal{Q}_R(i)) = R$ for $i \in \{3, \ldots, k-1\}$. Let $x, x' \in \mathcal{Q}_R(k)_0$ be step diagrams. Let $n, k_1, \ldots, k_n \geq 1$ be such that $k = k_1 + \cdots + k_n$. Suppose there exists $a, a' \in \mathcal{S}_R(n)_0$ and $b_i, b_i' \in \mathcal{S}_R(k_i)_0$ for each $i \in \{1, \ldots, n\}$ and $\sigma \in \Sigma_k$ such that $\sigma \circ \chi(a \otimes b_1 \otimes \cdots \otimes b_n) = x$ and $\sigma \circ \chi(a' \otimes b_1' \otimes \cdots \otimes b_n') = x'$. If $a \neq a'$ then there exists A such that $dA = a - a'$, since a and a' are homologous. Otherwise let $A = 0$. For each i, if $b_i \neq b_i'$ then there exists B_i such that $dB_i = b_i - b_i'$, since b_i and b_i' are homologous. Otherwise let $B_i = 0$. Then it is possible to choose signs judiciously to make the following equation hold:

$$d(A \otimes b_1 \otimes \cdots \otimes b_n + \sum_{i=1}^{n} {}^{+}_{-}(a' \otimes b_1' \otimes \cdots \otimes b_{i-1}' \otimes B_i \otimes b_{i+1} \otimes \cdots \otimes b_n))$$

$$= a \otimes b_1 \otimes \cdots \otimes b_n - a' \otimes b_1' \otimes \cdots \otimes b_n'$$

so that $a \otimes b_1 \otimes \cdots \otimes b_n$ and $a' \otimes b_1' \otimes \cdots \otimes b_n'$ are homologous. Thus x and x' are homologous.

For each $j \geq 2$, let $S_j =< 1, \ldots, j >\in \mathcal{S}_R(j)_0$. Then for any $n, k_1, \ldots, k_n \geq 1$ such that $k = k_1 + \cdots + k_n$, $\chi(S_n \otimes S_{k_1} \otimes \cdots \otimes S_{k_n}) = S_k$. Therefore $S_k \in \mathcal{Q}_R(k)$.

Let $\sigma \in \Sigma_k$. We show that $\sigma(S_k)$ is homologous to S_k. For each $r \geq 1$ and each $j \in \{1, \ldots, r-1\}$, let $T_j^r =< 1, \ldots, r-1, r, r-j >\in \mathcal{S}_R(r)_1$ (see figure 14). Then $dT_j^k = S {}^{+}_{-} \tau_j(S)$ where τ_j is the permutation $(k-j\ k-j+1 \ldots k) \in \Sigma_k$. Note that for each $a \neq k-j$:

$$(k-j\ k) =$$

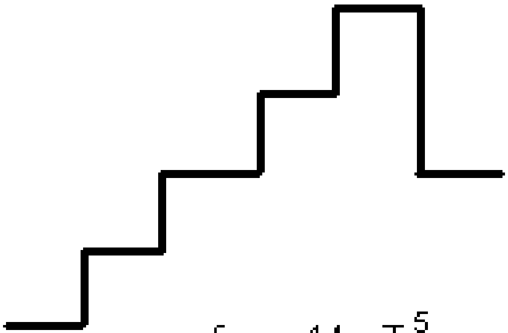


figure 14 - T_2^5

$$(k-1\ k)(k-2\ k-1\ k)\dots(k-j\ k-j+1\ \dots\ k)(k-j+1\ \dots\ k)\dots(k-1\ k)$$

and

$$(k-j\ a) = (a\ k)(k-j\ k)(a\ k)$$

so that any transposition in Σ_k is a product of the permutations τ_j and hence $\sigma = \tau_{j_1} \circ \dots \circ \tau_{j_s}$ for some $j_1, \dots, j_s$. Therefore by choosing signs judiciously,

$$d(\psi_{\tau_{j_s}} \circ \dots \circ \psi_{\tau_{j_2}}(T^k_{j_1}) \pm \psi_{\tau_{j_s}} \circ \dots \circ \psi_{\tau_{j_3}}(T^k_{j_2}) \pm \dots$$

$$\dots \pm \psi_{\tau_{j_s}}(T^k_{j_{s-1}}) \pm T^k_{j_s}) = S - \sigma(S)$$

so $\sigma(S_k)$ is homologous to S_k in $\mathcal{S}_R(k)$. To show that $\sigma(S_k)$ is homologous to S_k in $\mathcal{Q}_R(k)$, it remains to show that $T^k_j \in \mathcal{Q}_R(k)$. Let $I \in \mathcal{Q}_R(1)$ be the unique step diagram. Let $S =< 1, 2, 1 >\in \mathcal{Q}_R(2)_1$. Then $T^k_{k-1} = \chi(S \otimes S_{k-1} \otimes I)$ (see figure 15). Note that $T^k_j = \pm\chi(S_2 \otimes T^{j+1}_j \otimes S_{k-j-1})$ (see figure 16). Assuming, inductively, that $T^r_j \in \mathcal{Q}_R(r)$ for $r < k$, it follows that $T^k_j \in \mathcal{Q}_R(k)$.

Now, for any step diagram $S \in \mathcal{S}_R(k)_1$, $dS = A \pm B$ for step diagrams A and B. Therefore for any arbitrary $y \in \mathcal{S}_R(k)_1$, dy is not a step diagram. Hence, in particular, S_k does not bound (either in $\mathcal{S}_R(k)$ or $\mathcal{Q}_R(k)$). But

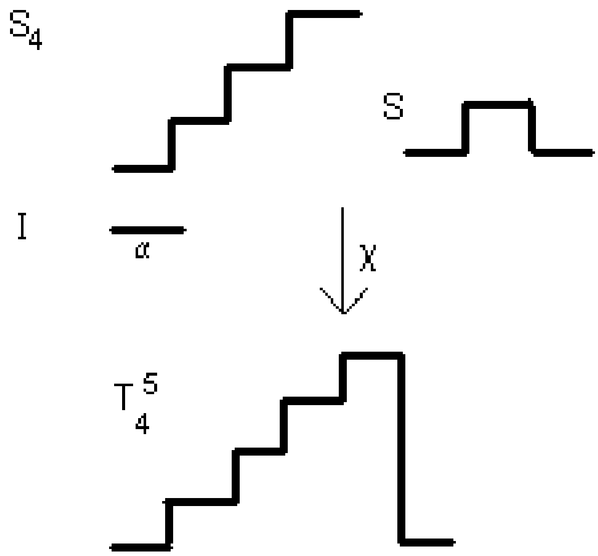


figure 15

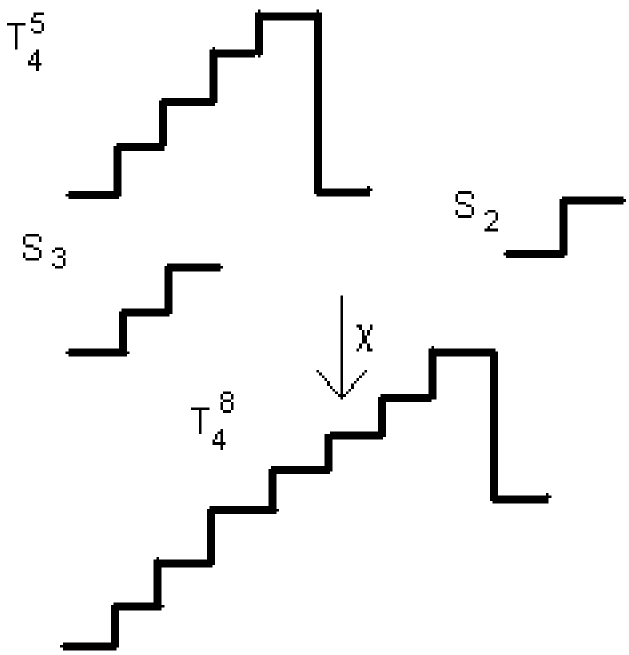


figure 16

since $dS_k = 0$, S_k represents a non-trivial homology class in both $\mathcal{S}_R(k)$ and $\mathcal{Q}_R(k)$. Therefore $H_0(\mathcal{Q}_R(k))$ is generated by the homology class of S_k, giving the required result. □

Lemma 6.4.4. *$\mathcal{Q}_R(k)$ admits a free action of Σ_k for every $k \geq 1$.*

Proof. $\mathcal{Q}_R(k)$ is spanned by $\chi(T \otimes S_1 \otimes \cdots \otimes S_n)$ for step diagrams $T \in \mathcal{S}_R(n)$, $S_1 \in \mathcal{S}_R(k_1), \ldots, S_n \in \mathcal{S}_R(k_n)$. Let $T \in \mathcal{S}_R(n)$, $S_1 \in \mathcal{S}_R(k_1), \ldots, S_n \in \mathcal{S}_R(k_n)$ and P, P' be two different T-partitions of $S_1, \ldots, S_n$. It suffices to note that for any $\sigma \in \Sigma_k$, $S(T; P) \neq \sigma(S(T; P'))$. Therefore $\sigma \circ \chi(T \otimes S_1 \otimes \cdots \otimes S_n) \neq 0$. □

We summarise the results of section 6.4 in the following theorem:

Theorem 6.4.5. *$\mathcal{Q}_R$ is a quadratic E_∞-operad.*

6.5 Steenrod's cup-i products

In the sequel, we will use $S^*(X)$ to denote $S^*(X; R)$. Since $S_*(X)$ is a coalgebra over $\mathcal{Q}_R$, it follows that $S^*(X)$ is an algebra over $\mathcal{Q}_R$ with structure maps

$$\theta^* : \mathcal{Q}_R(k) \otimes S^*(X)^{\otimes k} \to S^*(X)$$

$$\theta^*(S \otimes x_1 \otimes \cdots \otimes x_k)(y) = (x_1 \otimes \cdots \otimes x_k)(\theta(S \otimes y))$$

for $x_1, \ldots, x_k \in S^*(X)$ and $y \in S_*(X)$.

Let i be a non-negative integer. Let $S^i =< h_1, \ldots, h_{i+1} >$ be the step diagram of height 2 such that $h_{2a-1} = 2$ and $h_{2a} = 1$ for every a. Recall from [30] the definition of *i-regular*. Notice that two faces of a simplex are S^i-regular if and only if they are i-regular. The next theorem follows directly from Steenrod's definition of the $\cup_i$ product, as given in [30].

Theorem 6.5.1. *Let $x, y \in S^*(X)$. Then $x \cup_i y = \theta^*(S \otimes x \otimes y)$.*

Therefore the Steenrod cup-i products entirely determine the $\mathcal{Q}_R$-algebra structure of $S^*(X)$. The next theorem follows directly from this fact and [21, Main Theorem].

Theorem 6.5.2. *Let X and Y be finite type nilpotent spaces. Then X and Y are weakly equivalent if and only if there is a quasi-isomorphism $S^*(X;\mathbb{Z}) \to S^*(Y;\mathbb{Z})$ that commutes with all the cup-i products.*

Chapter 7

Space-like $\mathcal{S}_R$-coalgebras and homotopy groups

For the duration of this chapter, let $\mathbb{F}$ be a field. For brevity, denote $\mathcal{S}_{\mathbb{F}}$ by $\mathcal{S}$. Let C be an $\mathcal{S}$-coalgebra. Denote by C_n the n-th degree part of C.

Definition 7.0.1. Denote by S_n the unique step diagram of height 1, $n+2$ steps such that the left-hand step is of height 1. Denote by T_n the unique step diagram of height 1 with $n+2$ steps such that the left-hand step is of height 2. We define a map $M_n : C \to C \otimes C$ by $M_n(x) = \chi(S_n \otimes x)$.

Definition 7.0.2. C is said to be *space-like* if and only if, for each $n \geq 0$, $k > 0$, $x \in C_n$, we have $M_{n+k}(x) = 0$ and there exists a basis X_n for C_n such that for every $x \in X_n$, we have $M_n(x) = x \otimes x$.

Now suppose that C is space-like.

Definition 7.0.3. An element $x \in C_n$ is *primitive* if and only if $M_k(x) = 0$ for every $k < n$.

Definition 7.0.4. An element $x \in C_n$ is *group-like* if and only if $M_n(x) = x \otimes x$.

In the sequel, we will be using the simplicial cosimplicial set $\Delta[\cdot]$ given by the standard simplices. This is defined in [22, section 2] but for the purpose of notational clarity, the definition is outlined here.

Let Δ_k be the set $\{0, \ldots, k\}$ for each non-negative integer k. Let Δ be the category with objects Δ_k and morphisms $\mu : \Delta_n \to \Delta_m$ such that $\mu(i) \leq \mu(j)$ whenever $i < j$. Let $\partial_i : \Delta_m \to \Delta_{m+1}$ be given by $\partial_i(j) = j$ if $j < i$ and $\partial_i(j) = j + 1$ if $j \geq i$. Let $\tau_i : \Delta_m \to \Delta_{m-1}$ be given by $\tau_i(j) = j$ if $j \leq i$ and $\tau_i(j) = j - 1$ if $j > i$. Let $\Delta[n]$ be the simplicial set that in degree m is given by

$$\Delta[n]_m = Hom_{\Delta}(\Delta_m, \Delta_n)$$

and has face maps $d_i : \Delta[n]_m \to \Delta[n]_{m-1}$ given by $d_i(\mu) = \mu \circ \partial_i$ and degeneracies $s_i : \Delta[n]_m \to \Delta[n]_{m+1}$ given by $s_i(\mu) = \mu \circ \tau_i$. The various $\Delta[n]$ fit together to give a cosimplicial simplicial set $\Delta[\cdot]$ via the face maps $\delta_i : \Delta[n-1] \to \Delta[n]$ given by $\delta_i(\mu) = \partial_i \circ \mu$ and degeneracies $\sigma_i : \Delta[n+1] \to \Delta[n]$ given by $\sigma_i(\mu) = \tau_i \circ \mu$. Therefore the normalised chain complexes $C_*\Delta[n]$ (with coefficients in $\mathbb{F}$) fit together to give a cosimplicial chain complex $C_*\Delta[\cdot]$. It is simple to rephrase the results of chapters 5 and 6 for arbitrary simplicial sets rather than just the simplicial set given by the singular chains. Therefore $C_*\Delta[n]$ is an $\mathcal{S}$-coalgebra. Since δ_i and σ_i preserve the $\mathcal{S}$-coalgebra structure, $C_*\Delta[\cdot]$ is a cosimplicial $\mathcal{S}$-coalgebra. Let X_n^C be the set of $\mathcal{S}$-coalgebra morphisms from $C_*\Delta[n]$ to C. Then $X_\bullet^C$ is a simplicial set with face maps $D_i(\theta) = \theta \circ \delta_i$ and degeneracies $S_i(\theta) = \theta \circ \sigma_i$.

Let $x \in C_n$ be a non-zero group-like primitive element. Then since χ is a map of chain complexes, we have

$$S_{n-1}(x)^+_- T_{n-1}(x)^+_- S_n(\partial x)^+_-(\partial x \otimes x)^+_-(x \otimes \partial x) = 0.$$

Since x is primitive, $S_{n-1}(x) = T_{n-1}(x) = 0$. Since C is space-like, $S_n(\partial x) = 0$. Therefore

$$(\partial x \otimes x)^+_-(x \otimes \partial x) = 0.$$

By the bigrading of $C \otimes C$, it follows that $x \otimes \partial x = 0$ and since $x \neq 0$ by assumption, it follows that $\partial x = 0$. Therefore we can define a map $f_x : C_*\Delta[n] \to C$ as follows. Let $id_n : \Delta_n \to \Delta_n$ be the identity. Then define

$$f_x(id_n) = x$$

$$f_x(\mu) = 0$$

for every Δ-morphism $\mu : \Delta_m \to \Delta_n$ with $m \neq n$. This is a chain map since $C_m\Delta[n] = 0$ for $m > n$ and $\partial x = 0$ (since $C_m\Delta[n]$ has been normalised). It is a map of $\mathcal{S}$-coalgebras since x is primitive. Let $\tilde{X}^C_n$ be the set of all $\theta \in X^C_n$ such that $D_i(\theta) = 0$ for every i. Let GPC_n denote the set of group-like primitive elements in C_n.

Lemma 7.0.3. *The map $x \mapsto f_x$ defines an bijective map of sets*

$$F : GPC_n \to \tilde{X}^C_n.$$

Proof. The fact that F is injective is clear from the construction. We prove that it is surjective. Let $\theta \in \tilde{X}^C_n$. The task is to show that $\theta = f_{\theta(id_n)}$. Firstly, $D_i(\theta) = 0$ for every i so $\theta \circ \delta_i = 0$ for every i. Let $\mu : \Delta_{n-1} \to \Delta_n$ be a Δ-morphism. Then $\mu = \partial_i$ for some i so

$$\theta(\mu) = \theta(\partial_i) = \theta(\partial_i \circ id_{n-1}) = \theta \circ \delta_i(id_{n-1}) = 0.$$

A similar argument shows that if $\mu : \Delta_m \to \Delta_n$ is a Δ-morphism for $m < n$ then $\theta(\mu) = 0$ (this is proved by decomposing μ into a composition of face-maps). Notice that if $m < n$ then

$$M_m(\theta(id_n)) = (\theta \otimes \theta) \circ M_m(id_n) = \theta(\mu) \otimes \theta(\nu)$$

for Δ-morphisms μ, ν where at least one of μ, ν are of degree less than n in $C_*\Delta[n]$, so $\theta(\mu) \otimes \theta(\nu) = 0$. Hence $\theta(id_n)$ is primitive. Since $C_m\Delta[n]$ has been normalised, $\theta(\mu) = 0$ for $\mu \in C_m\Delta[n]$ for $m > n$. Note that

$$M_n(\theta(id_n)) = (\theta \otimes \theta) \circ M_n(id_n) = \theta(id_n) \otimes \theta(id_n)$$

so that $\theta(id_n)$ is group-like. Moreover, $\theta = f_{\theta(id_n)}$ so that F is surjective, hence bijective. □

Define an equivalence relation $\sim$ on GPC_n by $x \sim y$ if and only if the homology classes $[x]$ and $[y]$ in $H_n(C)$ are equal. Define an equivalence relation on $\tilde{X}_n^C$, also denoted $\sim$ (this ambiguity will be removed by the context), as per [22, definition 3.1]. The latter is Kan's construction of the homotopy groups of a simplicial set and so we denote

$$\pi_n(X_\bullet^C) := \frac{\tilde{X}_n^C}{\sim}.$$

Lemma 7.0.4. *The map F induces a bijection $F_* : \frac{GPC_n}{\sim} \to \pi_n(X_\bullet^C)$.*

Proof. Firstly, we show that this map is well-defined. Let $x, y \in GPC_n$ and suppose $x \sim y$. Then there exists some $z \in C_{n+1}$ such that $\partial z = x - y$. Consider the map $g : C_*\Delta[n+1] \to C$ given by $g(id_n) = z$, $g(\partial_n) = x$, $g(\partial_{n+1}) = y$, $g(\partial_i) = 0$ for $i < n$ and $g(\mu) = 0$ for every $\mu : \Delta_m \to \Delta_{n+1}$ for $m \neq n, n+1$. Then

$$D_n g(id_n) = g \circ \delta_n(id_n) = g(\partial_n \circ id_n) = g(\partial_n) = x$$

and $D_n g(\mu) = 0$ for every Δ-morphism $\mu : \Delta_m \to \Delta_n$ with $n \neq m$. Hence $D_n g = f_x$. Similarly, $D_{n+1} g = f_y$. It is easy to see that $D_i g = 0$ for every $i < n$. Therefore $f_x \sim f_y$. Hence F_* is well-defined. By running the previous argument backwards, we see that F_* is injective. F_* is surjective since F is. □

It is well-known that $\pi_n(X_\bullet)$ is a group for each n and any simplicial set $X_\bullet$. Hence there is a group structure on $\frac{GPC_n}{\sim}$ induced by F_*.

Definition 7.0.5. The *n-th homotopy group* of a space-like $\mathcal{S}$-coalgebra C is $\frac{GPC_n}{\sim}$. It is denoted $\pi_n(C)$.

Lemma 7.0.5. *$\pi_n(-)$ is a functor from space-like $\mathcal{S}$-coalgebras to groups.*

Proof. This is a simple direct check. □

Theorem 7.0.6. *Let X be a connected based topological space with basepoint $*$. Let $S_*(X)$ be the based singular chains on X with coefficients in $\mathbb{F}$. Then $S_*(X)$ is space-like and $\pi_n(S_*(X)) \cong \pi_n(X)$.*

Proof. The $\mathcal{S}$-coalgebra structure on $S_*(X)$ described in chapter 6 ensures that $M_{n+k}(x) = 0$ for every $k \geq 1$ and $x \in S_n(X)$. This structure also ensures every singular n-simplex x is sent to $x \otimes x$ by the map M_n. Since the singular n-simplices form a basis for $S_n(X)$, it follows that $S_*(X)$ is space-like.

The group-like elements of $S_n(X)$ are precisely the singular n-simplices. Let $\phi \in S_n(X)$ be a primitive singular n-simplex. For the purposes of this proof, we denote by Δ_n the geometric n-simplex with vertices $v_0, \ldots, v_n$. Then $\phi : \Delta_n \to X$. Suppose ϕ_1, ϕ_2 are S_{n-1}-regular singular sub-simplices of ϕ. Then either $\phi_1 = \phi \circ f$ or $\phi_2 = \phi \circ f$, where $f : \Delta_{n-1} \to \Delta_n$ is the inclusion of a face. If the former then $f = \partial_{2i}$ for some integer i. If the latter then $f = \partial_{2i+1}$ for some integer i. Both of these facts arise from the fact that ϕ_1, ϕ_2 are S_{n-1}-regular. It follows that

$$0 = M_{n-1}(\phi) = \sum_i {}^+_- \phi \otimes \partial_{2i+1}\phi + \sum_i {}^+_- \partial_{2i}\phi \otimes \phi$$

since ϕ is primitive. If ϕ is non-degenerate then $\partial_i \phi = 0$ for each i.

Conversely, suppose ϕ is a singular n-simplex such that $\partial_i\phi = 0$ for each i. Let $k < n$. Then $M_k(\phi) = \partial_{i_1} \dots \partial_{i_s}\phi \otimes \partial_{j_1} \dots \partial_{j_{k-s}}\theta$ for some $i_1, \dots, i_s, j_1, \dots, j_{k-s}$ and $0 \le s \le k$. Hence $M_k(\phi) = 0$. Hence the group-like primitive elements of $S_*(X)$ are singular n-simplices ϕ that are either degenerate or have $\partial_i\phi = 0$ for each i; and each n-simplex with the latter property is primitive and group-like.

If ϕ is a degenerate primitive singular n-simplex then it will always be null-homologous. So let ϕ_1, ϕ_2 be non-degenerate primitive singular n-simplices. Suppose there is some $\theta \in S_{n+1}(X)$ such that $\partial\theta = \phi_1 - \phi_2$. Then $\theta = \sum_{k=1}^{N} \theta_k$ where θ_k are singular $n+1$-simplices. The union of the images of the various θ_k, $\bigcup_k im(\theta_k)$ is connected. Therefore we can view θ as a map from a connected geometric simplicial complex K to X such that all the simplices on the boundary of K map to the base-point $*$ except for two: one of which maps to the image of ϕ_1 and the other to the image of ϕ_2. Therefore we can define a map $\theta' : \Delta_{n+1} \to X$ such that each of the vertices of Δ_{n+1} are sent to p, the n-face $\partial_n\Delta_{n+1}$ is sent to $\partial_n\theta_1$, the n-face $\partial_{n+1}\Delta_{n+1}$ is sent to $\partial_{n+1}\theta_N$ and the image of θ' is the same as the image of $\theta : K \to X$. It follows that ϕ_1 and ϕ_2 are equivalent elements of the simplicial set given by the singular simplices in the sense of [22, definition 3.1]. Hence the homology class of ϕ_1 represents a unique homotopy class of X (see [22, theorem 16.1]).

Conversely, every homotopy class of X is represented by a singular simplex ϕ with $\partial_i\phi = 0$ for each i. Two such singular n-simplices x and y are homotopic if and only if there is some singular simplex z such that $\partial_n z = x$, $\partial_{n+1} z = y$ and $\partial_i z = 0$ for $i \neq n, n+1$. Since z is an element of $S_*(X)$ and $\partial z = x - y$, it follows that x and y are homologous elements of $S_*(X)$. Since x and y are also primitive and group-like, it follows that the set $\pi_n(S_*(X)) = \frac{GPS_n(X)}{\sim}$ is precisely the set $\pi_n(X)$. The fact that their group structures are the same comes from the fact that the group structure on $\pi_n(S_*(X))$ is induced by the group structure on $\pi_n(X_\bullet^{S_*(X)})$ and the map F induces a homomorphism from $\pi_n(X)$ (where $\pi_n(X)$ is Kan's construction of the homotopy groups of the simplicial set of singular chains on X) to $\pi_n(X_\bullet^{S_*(X)})$. $\square$

Theorem 7.0.7. *Let C, D be space-like $\mathcal{S}$-coalgebras. Suppose there is a quasi-isomorphism of $\mathcal{S}$-coalgebras $f : C \to D$. Then $\pi_n(f) : \pi_n(C) \to \pi_n(D)$ is injective.*

Proof. This is a simple, direct check. $\square$

Bibliography

[1] A. Bousfield and D. Kan, *Homotopy limits, completions and localisations*, Lecture notes in maths 304, Springer (1972)

[2] R. L. Cohen and J. D. S. Jones, *A homotopy theoretic realization of string topology*, Math. Ann. 324 (2002) pp. 773-798.

[3] R. L. Cohen, J. D. S. Jones and J. Yan, *The loop homology algebra of spheres and projective spaces*, Categorical decomposition techniques in algebraic topology (Isle of Skye 2001) pp. 77-92 Progr. Math 215

[4] R. Cohen, K. Hess, A. Voronov and D. Sullivan, *String topology and cyclic homology*, Advanced Courses in Mathematics CRM Barcelona S, Birkhauser Verlag (2004)

[5] M. Chas and D. Sullivan, *String topology*, preprint: math.GT/9911159 (1999)

[6] P. Deligne, P. Griffiths, J. Morgan and D. Sullivan *Real homotopy theory of Kähler manifolds*, Inventiones Math 29 (1975) pp. 245-274

[7] Felix, Halperin and Thomas, *Rational homotopy theory*, Springer (2001)

[8] Y. Felix, J-C. Thomas and M. Vigue-Poirrier, *Loop homology algebra of a closed manifold*, preprint math.AT/0203137 (2002)

[9] Y. Felix, J-C. Thomas and M. Vigue-Poirrier, *Rational String topology*, preprint math.AT/0406593 (2004)

[10] E. Getzler and J. D. S. Jones, *Operads, homotopy algebra and iterated integrals for double loop spaces*, preprint hep-th/9403055 (1994)

[11] Ginzburg and Kapranov, *Koszul duality for operads*, Duke maths journal 76 (1994) pp. 203-272

[12] A. Hatcher *Algebraic topology*, Cambridge Univ Press (2002)

[13] Hinich and Schechtmann, *On the homotopy limit of homotopy algebras*, K-theory, arithmetic and geometry, Lecture notes in maths 1289, Springer (1987) pp. 240-264

[14] J. D. S. Jones, *Lectures on operads*, Contemporary Maths 315 (2002) pp. 89-130

[15] T. V. Kadeichvili, *On the homology theory of fibre spaces*, Russian Maths Surveys 6 (1980) pp. 231-238

[16] M. Kontsevich, *Deformation Quantisation of Poisson manifolds, I*, preprint math.q-alg/9709040 (1997)

[17] I. Kriz, J. P. May, *Operads, algebras, modules and motives*, Asterisque 233 (1995)

[18] Loday, *Cyclic Homology*, Springer (1998)

[19] T. Lada and M. Markl, *Strongly homotopy Lie algebras*, Comm. algebra 23 (1995) pp. 2147-2161

[20] M. Mandell, *E_∞-algebras and p-adic homotopy theory*, Topology 40 (2001) pp. 43-94

[21] M.Mandell *Cochains and homotopy type*, preprint math.AT/0311016 (2003)

[22] J. P. May, *Simplicial Objects in Algebraic Topology*, Univ. of Chicago Press (1967)

[23] J. P. May, *A general algebraic approach to Steenrod operations*, The Steenrod algebra and its applications, Lecture notes in maths 168, Springer (1970)

[24] M. Markl, S. Schnider and J. Stasheff, *Operads in algebra, topology and physics*, Mathematical Survays and Monographs 96, AMS publications (2002)

[25] J. McCleary, *A users guide to spectral sequences*, Cambridge studies in advanced mathematics 58 (2001)

[26] J. Milnor and J. Stasheff *Characteristic classes*, Annals of Maths studies 76, Princeton University Press (1974)

[27] D. Quillen, *Rational homotopy theory*, Annals of maths 90 (1969) pp. 205-295

[28] A. Robinson, S. Whitehead, *Operads and* Γ*-homology of commutative rings*, Math. Proc. Camb. Philos. Soc. 132 (2002) pp.197-234

[29] J. R. Smith, *Operads and Algebraic Homotopy*, preprint: math.AT/0004003

[30] N. E. Steenrod, *Products of cocycles and extensions of mappings*, Annals of Maths 48 (1947) pp. 290-320

[31] D. Sullivan, *Genetics of Homotopy theory and the Adams conjecture*, Annals of Maths 100 (1974) pp. 1-79

[32] D.Sullivan, *Infinitesimal computations in topology*, I.H.E.S. publications 47 (1977) pp. 269-331

[33] T. Tradler, *Poincare Duality Induces a BV-structure on Hochschild Cohomology*, CUNY PhD thesis (2002)